Long-term risk with stochastic interest rates*

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Abstract

Investors with heterogeneous trading horizons require compensation for the exposure to different risks. The no-arbitrage valuation over increasing horizons is described by the evolution of stochastic discount factors (SDFs). Each of them exhibits a multiplicative decomposition into deterministic growth term, permanent and transient component, provided by Hansen and Scheinkman (2009). In particular, the growth rate captures the deterministic discounting for risks that are relevant in the long term. When interest rates in the market are constant, the SDF growth rate coincides with the instantaneous rate. On the contrary, when rates of interest are stochastic, the SDF growth rate is given by the long-term yield of zero-coupon bonds, which is unsuitable for instantaneous no-arbitrage valuation.

We show how to reconcile the long-run properties of the SDF with the instantaneous relations between returns and rates in stochastic-rate markets. In particular, we introduce a rate adjustment in pricing that isolates the short-term variability of rates. No-arbitrage prices are then factorized into rate-adjusted prices and a rate adjustment that is absent when interest rates are constant. Rate-adjusted prices employ constant yields to maturity for discounting future payoffs over time. The rate-adjusted SDF features the same long-term growth rate of the SDF in the market but has no transient component in its Hansen-Scheinkman decomposition. Therefore, rate-adjusted prices provide the proper valuation for long-term interest rate risk. Moreover, we show how this novel notion is fruitful for managing the interest rate risk related to fixed-income derivatives, life insurances and annuities.

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1 Motivations and main results

A deep analysis of the term structure of interest rates is essential for dynamic asset pricing theory. Once fixed-income securities are issued by an institution, they are likely to display time-varying yields to maturity in the secondary market. In fact, different yields embody the changeable expectations of market participants about future events that could impinge on the credit market at different horizons. Moreover, the relation between short- and long-term bond yields is an indicator of the overall performance of the economy. For instance, inverted yield curves may constitute a predictor for recessions (Ang et al., 2006). In general, understanding the consistent aggregation of interest rate risks associated with heterogeneous maturities has been a challenge of the last century. A substantial stream of economic literature involves the expectation theory, introduced by Fisher (1930) and developed by Keynes (1930). Subsequent elaborations are due to Lutz (1940) and Hicks (1946), among the others. See the surveys by Cox et al. (1985) and Russell (1992).

Similarly, in dynamic asset pricing theory, investors with different trading horizons require compensation for the exposure to sources of randomness with heterogeneous duration. As a consequence, long-term premia are the outcome of the intertemporal aggregation of local risk exposures. Importantly, the components of economic variables that survive in the long run are free from inaccuracies due to temporary risk adjustments and are likely to reflect economic fundamentals. For example, Hansen et al. (2008) provide a theoretical analysis of risk-return trade-offs across increasing maturities.

Hansen and Scheinkman (2009) develop a general operator framework to connect short- and long-run risk compensations. Their valuation of cashflows exploits the evolution of stochastic growth and stochastic discount functionals. In particular, in arbitrage-free markets every stochastic discount factor features a multiplicative decomposition in a deterministic growth term, a permanent (or martingale) component and a transient factor. The martingale component induces a probability measure change in the long run and is responsible for a large variance explanation. In equilibrium this decomposition impacts the asymptotic dynamics of aggregate consumption and wealth through investors' marginal utility: Alvarez and Jermann (2005) and Hansen (2012).

Stochastic discount factors reflect the risk-neutral valuation of state-contingent payoffs in viable markets. In particular, the deterministic growth rate elicited by the Hansen-Scheinkman decomposition provides a proper discounting adjusted for risks that are relevant in the long run. When interest rates in the market are constant over time, the pricing kernel growth rate coincides with the same instantaneous interest rate at any maturity under scrutiny. When rates are stochastic, their variability is captured by the interplay of stochastic discounting and stochastic growth, as discussed in Borovička et al. (2011). Unfortunately, in this case it is hard to achieve a synthetic characterization of pricing kernel growth rates across increasing horizons.

This work fills this gap and builds a bridge between fixed-income and long-term risk literatures, showing the extent to which the intertemporal aggregation of instantaneous random rates contributes to pricing kernel growth rates at increasingly large horizons. Moreover, by shifting the maturity farther and farther over time, we show how to retrieve the asymptotic deterministic growth rate inferred by Qin and Linetsky (2017). Notably, our results do not depend on the specific dynamics of interest rates.

1.1 Summary of main results

We consider a continuous-time arbitrage-free market in the time interval $[0, T]$, composed of both risky and fixed-income securities that depend on a stochastic instantaneous rate Y_t. We concentrate on price dynamics, while simultaneously focusing on the evolution of stochastic discount factors. We develop the theory in a conditional setting.

Regarding price dynamics, we start considering the no-arbitrage price $\pi_t(h_T)$ at time $0 \leqslant t \leqslant T$ of an attainable payoff paid at T. When interest rates are constant over time, the absence of arbitrages leads to the equality

$$\text{instantaneous asset return} = \text{instantaneous risk-free rate}$$

under a risk-neutral measure Q. Hansen and Scheinkman (2009) and Hansen (2012) formalize this relation as an eigenvalue-eigenvector problem for the extended infinitesimal generator in a Markov setting. Marinacci and Severino (2018) formulate the same eigenvalue-eigenvector problem in a more general framework (without requiring the Markov property):

$$\mathcal{D}_0 \pi_t = r \ \pi_t, \qquad t \in [0, T]. \tag{1}$$

Here, $\mathcal{D}_0$ is the weak time-derivative, a differential operator that applies to a wide class of semimartingales and generalizes the infinitesimal generator. Indeed, weak time-differentiability can be investigated for any adapted process $u : [0, T] \to L^1$ that is L^1-right-continuous in $[0, T)$, L^1-left-continuous at T and has finite $\int_0^T \mathbb{E}[|u_\tau|]d\tau$. See details in Marinacci and Severino (2018). Eq. (1) generalizes the ordinary differential equation satisfied by risk-free bonds with interest rate r and this eigenvalue-eigenvector problem captures the absence of arbitrage. Indeed, asking $\mathcal{D}_0\pi_t = r\ \pi_t$ is equivalent to requiring $\mathcal{D}_0(e^{-rt}\pi_t) = 0$, i.e. discounted no-arbitrage prices are Q-martingales (Proposition 3.1 in Marinacci and Severino, 2018). Importantly, the eigenvalue r has a prominent role in the evolution of the stochastic discount factor since it captures its deterministic growth rate. Indeed, the pricing kernel in any time interval $[s,t]$ is $M_{s,t} = e^{-r(t-s)}L_{s,t}$, where $L_{s,t}$ is the conditional Radon-Nikodym density of Q with respect to the physical probability P.

When rates are constant over time, the instantaneous rate determines both the risk-neutral price dynamics and the pricing kernel growth rate, irrespective of the time interval under consideration. In a stochastic-rate setting, the action of instantaneous rates is more convoluted. Indeed, while eq. (1) may be rephrased at any instant t by using the random rate Y_t instead of r, the sole instantaneous rate Y_t is unable to subsume the stochastic discount factor growth rate on any given time period. In particular, the pricing kernel in $[s,t]$ takes the form $M_{s,t} = e^{-\int_s^t Y_\tau d\tau}L_{s,t}$.

Comparing the expressions of $M_{s,t}$ in the two contexts, we obtain an indication of the heavy measurability requirement of the growth term $e^{-\int_s^t Y_\tau d\tau}$ with respect to the deterministic $e^{-r(t-s)}$. The definition of a growth rate for $M_{s,t}$ is indeed challenging when interest rates are floating over time. Under regularity conditions, Qin and Linetsky (2017) individuate a deterministic *long-term yield* as the asymptotic growth rate of the stochastic discount factor. Nevertheless, the quantity identified by Qin and Linetsky – which is the limit of bonds yields at infinite horizons – is not suitable for characterizing instantaneous returns in the sense of eq. (1). The conceptual reason beyond these difficulties rests upon the dual nature of stochastic interest rates. On the one hand, they represent a proxy for the (absent) risk-free rates. On the other, they constitute a source of randomness per se.

The main contribution of this paper is to provide a generalization of eq. (1) for stochastic-rate settings where the employed eigenvalue is determinant for the pricing kernel growth rate. The previously described relations for rates that are constant over time follow as a special case of our construction. In addition, our generalization is consistent with the asymptotic results of the long-term risk literature.

Our theory is based on the introduction of *rate-adjusted prices* that provide hedging from interest rate variability. In general, given an attainable payoff h_T at time T, we denote by $\pi_t(h_T)$ its no-arbitrage price at t. For example, $\pi_t(1_T)$ is the no-arbitrage price of a zero-coupon T-bond (T-ZCB for brevity). The rate-adjusted price of h_T at time t in the interval $[s,T]$, denoted by

$\rho_t^T(s, h_T)$, satisfies

$$\underbrace{\rho_t^T\left(s, h_T\right)}_{rate-adjusted\ price} = \underbrace{e^{r_s^T(t-s)} \frac{\pi_s\left(1_T\right)}{\pi_t\left(1_T\right)}}_{adjustment} \cdot \underbrace{\pi_t\left(h_T\right),}_{no-arbitrage\ price} \tag{2}$$

where r_s^T is the yield to maturity of a T-ZCB at time s. If interest rates are constant, rate-adjusted and no-arbitrage prices coincide. In general, the two prices are equal at the instants s and T.

Interestingly, ρ^T can be interpreted as an indifference price for an investor that ignores the variability of rates, or as the conversion price of a convertible bond (when h_T refers to a stock). In particular, the rate-adjusted price of a ZCB is the no-arbitrage price of the same security in a parallel market with constant interest rates equal to the yield r_s^T. Moreover, when the time horizon T goes to infinity, the ratio between ρ_t^T and π_t is convergent both in probability and in expectation.

From the standpoint of stochastic discount factors, we define the *rate-adjusted pricing kernel* in the interval $[s, t]$ with $s \leqslant t \leqslant T$ by

$$N_{s,t}^T = e^{-r_s^T(t-s)} \frac{\pi_t\left(1_T\right)}{\pi_s\left(1_T\right)}\ M_{s,t}.$$

As expected, $N_{s,t}^T$ and $M_{s,t}$ coincide when interest rates are constant.

To formalize our differential pricing relations in a conditional setting, we introduce a mathematical instrument that allows us to differentiate stochastic processes on the time window $[s, T]$ by disentangling the role of known information at instant s: the *weak time-derivative in* $[s, T]$, denoted by $\mathcal{D}_s$, which is a generalization of the weak time-derivative of Marinacci and Severino (2018). Similarly to the original notion, the weak time-derivative in $[s, T]$ applies to a large class of adapted processes (without requiring the Markov property), ensuring a great generality of the results. Moreover, the weak time-derivative in $[s, T]$ provides a handy characterization of conditional martingales, allowing us to easily deal with discounted no-arbitrage prices and forward prices. These processes feature, indeed, a null weak time-derivative in $[s, T]$, which conveniently generalizes the zero-drift condition of continuous-time asset pricing to a wide class of semimartingales. The absence of requirements on the Markov property makes our framework compatible with Qin and Linetsky (2017) and allows our theory to embrace the results therein. An application focused on a non-Markov setting is illustrated in Qin and Linetsky (2018).

Our main results are Theorems 2 and 8. We prove that rate-adjusted prices and rate-adjusted pricing kernels satisfy the differential relations

$$\mathcal{D}_s \rho_t^T = r_s^T \rho_t^T, \qquad \mathcal{D}_s N_{s,t}^T = -r_s^T N_{s,t}^T \tag{3}$$

for any t in $[s, T]$, where the first equality holds under the forward measure and the second one under the physical measure. These equations parallel the relations satisfied by no-arbitrage prices and by the pricing kernel $M_{s,t}$ when interest rates are constant:

$$\mathcal{D}_s \pi_t = r \pi_t, \qquad \qquad \mathcal{D}_s M_{s,t} = -r M_{s,t},$$

where the two equalities hold under the risk-neutral measure Q and the measure P, respectively. The equations on the left rephrase the relation between returns and rates, while the parameters in the right-hand side equalities identify the pricing kernel growth rates.

The constant rate is replaced by the yield r_s^T, conveying the intuition that ZCBs play the role of risk-free assets in a context in which the money market account is unforeseeable. The results are robust when the time horizon is moved to infinity (Subsections 3.5 and 4.2): the long-term yield arises as the growth rate for both the rate-adjusted pricing kernel and $M_{s,t}$. Indeed, the long-term rate-adjusted pricing kernel $N_{s,t}^{\infty}$ differs from $M_{s,t}$ only in the transient component in the Hansen and Scheinkman (2009) decomposition. Such component is trivial only for $N_{s,t}^{\infty}$ when interest rates are stochastic. So, pricing with the rate adjustment means overlooking the transient component in the pricing kernel and, hence, focusing on the long-term interest rate risk.

The dynamics of eq. (3) are conditional on the information structure available at date s. We are actually considering an eigenvalue-eigenvector problem in which the eigenvalue is a random variable, known at the beginning of the trading interval. This approach combines the intuition of conditional asset pricing theory – that dates back to Hansen and Richard (1987) – with the Perron-Frobenius theory applied by Hansen and Scheinkman (2009).

Results similar to eq. (3) are unlikely to hold for no-arbitrage prices when rates are stochastic: the instantaneous rate Y_t cannot serve as eigenvalue and the transitory component of $M_{s,t}$ hinders the differentiation. Nonetheless, rate-adjusted prices are able to reach the purpose. Indeed, ZCB yields play both the role of eigenvalues (for rate-adjusted prices) and that of growth rates (for rate-adjusted pricing kernels), generalizing the features of the constant-rate case. Table 1 summarizes our results.

As we show in Propositions 4 and 7, when T goes to infinity, the adjustment in eq. (2) converges to the transient component of $M_{s,t}$ and, for this reason, the rate-adjusted pricing kernel $N_{s,t}^{\infty}$ turns out to have a trivial transient component. As a result, the adjustment can be associated with temporary rate fluctuations and eq. (2) is actually a decomposition of the no-arbitrage price into a rate-adjusted price (sensible to long-lasting shocks in the yield curve) and the adjustment (sensible to temporary rate variations). We are, then, providing a way to disentangle the short- and long-term interest rate risks of a given cashflow. Therefore, our theory has promising applications to the risk management of financial and insurance products with distant maturities and significant exposure to interest rate variability. In Section 5 we provide a bundle of applications of rate-adjusted valuation to derivative pricing (interest rate swaps), life insurance actuarial present values, mortgage renegotiations and pension fund withdrawals. Section 6 illustrates the theory in the context of single-factor affine interest rate models. The Appendix discusses several theoretical issues and provides all the proofs.

2 Asset pricing framework

We start with describing a general continuous-time arbitrage-free market with stochastic interest rates. We then focus on the conditions that ensure long-term convergences, describe the forward measures and discuss the main properties of weak time-derivatives. All technical details are collected in Appendix A.

2.1 Arbitrage-free market

We fix the time horizon $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and is left-continuous in T. We call P the *physical measure*. An adapted random process $Y = \{Y_t\}_{t\in[0,T]}$ represents the instantaneous interest rate and $\{e^{\int_0^t Y_\tau d\tau}\}_{t\in[0,T]}$ is the money market account. ZCBs with any possible maturity (and face value of one unit) are traded. Additional risky securities, with adapted price processes, can be present in the market, too.

If $X = \{X_t\}_{t\in[0,T]}$ denotes the price process of a traded security, at any instant t its relative price $\tilde{X}_t = e^{-\int_0^t Y_\tau d\tau} X_t$ is obtained by discounting with the money market account. We assume that all relative asset prices are semimartingales and there exists an equivalent martingale measure Q (the *risk-neutral measure*). Hence, arbitrage opportunities are ruled out. We indicate by L_T the Radon-Nikodym derivative of Q with respect to P on $\mathcal{F}_T$, i.e. $L_T = dQ/dP$, and we set $L_t = \mathbb{E}_t[L_T]$ for all $t \in [0,T]$, where $\mathbb{E}_t$ is the conditional expectation with respect to $\mathcal{F}_t$ under P. We also define $L_{t,T} = L_T/L_t$.

We denote by $M = \{M_t\}_{t\in[0,T]}$ the strictly positive stochastic discount factor process associated with Q, namely $M_t = e^{-\int_0^t Y_\tau d\tau} L_t$ for all t. In addition, we define the pricing kernel in any time interval $[s,t]$ with $s \leqslant t \leqslant T$ by $M_{s,t} = M_t/M_s = e^{-\int_s^t Y_\tau d\tau} L_{s,t}$.

The no-arbitrage price at time t of a T-ZCB is $\pi_t(1_T)$ and the related *yield to maturity* is

$$r_t^T = -\frac{\log \pi_t(1_T)}{T-t} = -\frac{\log \mathbb{E}_t^Q\left[e^{-\int_t^T Y_\tau d\tau}\right]}{T-t}.$$

We also define r_T^T as the a.s. limit of r_t^T when t approaches T. The yield r_t^T may be interpreted as an average interest rate which is ex-ante equivalent to the compounding of all forthcoming instantaneous rates Y_τ when τ spans the interval $[t, T]$. Clearly, if interest rates are constant over time (and deterministic), the yield coincides with the instantaneous rate.

The time horizon T is fixed in several sections of the paper while in other parts we move the horizon T to infinity (Subsections 2.2, 2.3, 3.4, 3.5, 4.1, 4.2, 5.2 and 6.1.2). When T increases, any payoff h_T paid at time T also moves farther in time and, in particular, the T-ZCB features a farther maturity. This case is considered, for example, in Subsections 3.4 and 6.1.2. However, in Subsection 3.5, the payment date is disentangled from the time horizon T and only the latter moves to infinity. This approach permits to illustrate, in Subsection 3.6, rate-adjusted pricing for a given cashflow where only the last payment may coincide with the time horizon T.

2.2 Long-term assumptions

Our semimartingale framework is compatible with the setting of Qin and Linetsky (2017) that formalize the convergence of bond yields, forward measures and stochastic discount factors when the horizon T becomes larger and larger.

Assumptions 1 *We assume the following.*

- *M_t is a strictly positive semimartingale such that $\mathbb{E}[M_{t,T}]$ is finite for all $0 \leqslant t < T$.*
- *For all $t > 0$, when T goes to infinity, $\mathbb{E}_t[M_T]/\mathbb{E}[M_T]$ converges in L^1 to a positive $\mathcal{F}_t$-measurable random variable G_t^∞.*
- *For all $t > 0$, when T goes to infinity, $\pi_0(1_{T-t})/\pi_0(1_T)$ has a positive finite limit.*
- *For all $t > 0$, when T goes to infinity, the limit in probability of r_t^T is positive.*

The three first assumptions are from Qin and Linetsky (2017) and ensure that, for any t, the yield to maturity r_t^T converges in probability to a deterministic *long-term yield* r^∞ (Theorem 3.2 therein). This result is consistent with the persistence of the yield curve over large maturities (Diebold and Li, 2006). Notably, r^∞ is not dependent on t, consistently with the impossibility of falling long-term rates (Dybvig et al., 1996). The fourth assumption additionally ensures that r^∞ is positive.

The long-term yield is associated with a long bond, obtained by a roll-over portfolio of ZCBs across increasing maturities. The time t value of this portfolio is denoted by B_t^∞, while b_t^∞ is its long-term discounted value: $b_t^\infty = e^{-r^\infty t} B_t^\infty$.

2.3 Forward measures and pricing

By using as numéraire the no-arbitrage price of a T-ZCB, we construct the forward measure with horizon T or, simply, *T-forward measure* and we denote it by F^T (Geman et al., 1995). This probability measure is equivalent to Q and we indicate its Radon-Nikodym derivative with respect to P on $\mathcal{F}_T$ by $G_T^T = dF^T/dP$. Moreover, we set $G_t^T = \mathbb{E}_t[G_T^T]$ for any $t \in [0, T]$ and we define $G_{t,T}^T = G_T^T / G_t^T$. In the special case in which interest rates are constant over time, F^T coincides with Q. Using G_t^T, the stochastic discount factor and the pricing kernel in any interval $[s, t]$ may be expressed as

$$M_t = e^{r_t^T (T-t) - r_0^T T} G_t^T, \qquad M_{s,t} = e^{r_t^T (T-t) - r_s^T (T-s)} G_{s,t}^T.$$

Although $M_{s,t}$ refers to the time window $[s, t]$, its expression depends on the ultimate horizon T through the density of the T-forward measure.

The T-forward measure provides a handy representation of no-arbitrage prices. Consider, for instance, an attainable $\mathcal{F}_T$-measurable payoff h_T such that $\mathbb{E}_s^{F^T}[|h_T|]$ is $\mathcal{F}_s$-measurable. Its no-arbitrage price at any time $t \in [s, T]$ can be written in equivalent ways, depending on the numéraire:

$$\pi_t(h_T) = \mathbb{E}_t^Q \left[e^{-\int_t^T Y_\tau d\tau} h_T \right] = e^{-r_t^T (T-t)} \mathbb{E}_t^{F^T} [h_T]. \tag{4}$$

The right-hand side of this expression actually makes a fruitful bridge between constant-rate and stochastic-rate asset valuation.

We now consider the case in which the horizon T becomes arbitrarily large under the Assumptions 1. Qin and Linetsky (2017) prove in Theorem 3.1 that, on each $\mathcal{F}_t$, F^T strongly converges to the *long-term forward measure* F^∞ when T goes to infinity. At any time t, we denote the Radon-Nikodym derivative of F^∞ with respect to P by $G_t^\infty = dF^\infty/dP|_{\mathcal{F}_t}$ and the collection of all G_t^∞ constitutes a P-martingale. In addition, we set $G_{s,t}^\infty = G_t^\infty / G_s^\infty$ for all s and t. Interestingly, the long-term forward measure is related to a specific numéraire: the price B_t^∞ of the long bond.

Finally, Qin and Linetsky (2017) prove in Theorem 3.2 that the pricing kernel $M_{s,t}$ satisfies the long-run decomposition

$$M_{s,t} = e^{-r^{\infty}(t-s)} \ \frac{b_s^{\infty}}{b_t^{\infty}} \ G_{s,t}^{\infty}. \tag{5}$$

In a Markov environment, $G_{s,t}^{\infty}$ defines the *permanent* (or *martingale*) *component* of Hansen and Scheinkman (2009) decomposition, while $b_s^{\infty}/b_t^{\infty}$ constitutes the *transient component* and r^{∞} is the deterministic *long-term growth rate*. For the empirical estimation of these terms see Christensen (2017) and Qin et al. (2018). Furthermore, from eq. (5) it is easy to write the no-arbitrage price of any payoff h_T by using the long-term forward measure:

$$\pi_t\left(h_T\right) = e^{-r^{\infty}(T-t)} \mathbb{E}_t^{F^{\infty}}\left[\frac{b_t^{\infty}}{b_T^{\infty}} h_T\right]. \tag{6}$$

2.4 Weak time-derivative in $[s, T]$

To analyze the increments of stochastic processes we need a differential operator. Marinacci and Severino (2018) introduce the *weak time-derivative* that applies to a wide class of semimartingales. Here we extend this notion to a conditional setting. Appendix A collects the most technical details and the proofs.

We fix an instant $s \in [0, T]$ and we consider the conditional space $L_s^1(\mathcal{F}_T)$ composed of variables $f \in L^0(\mathcal{F}_T)$ such that $\mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)$ and endowed with the L^0-valued metric $d(f,g) = \mathbb{E}_s[|f-g|]$. Moreover, we denote by $\mathcal{U}_s$ the L^0-module of adapted processes $u : [s,T] \to L_s^1(\mathcal{F}_T)$ that are L_s^1-right-continuous in $[s, T)$ and L_s^1-left-continuous at T. We then define the weak time-derivative for processes in $\mathcal{U}_s$. For any $t \in [s,T]$ the definition uses the space $C_c^1((t,T), L^0(\mathcal{F}_s))$ of functions $\varphi_s : [t, T] \to L^0(\mathcal{F}_s)$ that have compact support in (t, T) and are continuously differentiable over time. The integrals in the next definition are pathwise integrals of processes in $L^0(\mathcal{F}_s)$.

Definition 1 *We say that a process $u \in \mathcal{U}_s$ is* weakly time-differentiable in $[s,T]$ *when there exists a process $v \in \mathcal{U}_s$ such that for every $t \in [s,T]$*

$$\int_t^T \mathbb{E}_s\left[v_\tau \mathbf{1}_{A_t}\right] \varphi_s(\tau) d\tau = -\int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] \varphi_s'(\tau) d\tau$$

for all $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$. In this case, we call v a weak time-derivative *of u in $[s,T]$.*

Definition 1 generalizes the weak time-derivative of Definition 2.1 in Marinacci and Severino (2018), where $s = 0$ and deterministic test functions are employed. The weak time-derivative in $[s,T]$ is unique (Proposition 10 in Appendix A) and we denote it by $\mathcal{D}_s u$. Then, we introduce the L^0-submodules of $\mathcal{U}_s$, denoted by $\mathcal{U}_s^1$ and $\mathcal{U}_s^\infty$, that consist of weakly time-differentiable (or infinitely weakly time-differentiable) processes in $[s, T]$.

As shown in Marinacci and Severino (2018), the notion of weak time-derivative applies to any adapted process (with few regularity conditions), differently from the infinitesimal generator (requiring the Feller property) and the extended infinitesimal generator (requiring the Markov property). However, when these properties hold, we retrieve the infinitesimal generator and its extended version. Compared to these alternatives, the advantage of using weak time-derivatives consists in the larger class of processes to consider, which ensures a greater generality of the results.

As to weak time-derivative in $[s, T]$, it is important to notice that $\mathcal{F}_s$-measurable functions play the role of multiplicative constants. Given a process $u \in \mathcal{U}_s^1$ and $\xi_s \in L^0(\mathcal{F}_s)$, the process defined by $\xi_s u_t$ for all $t \in [s, T]$ belongs to $\mathcal{U}_s^1$, too, and $\mathcal{D}_s(\xi_s u) = \xi_s \mathcal{D}_s u$. This property permits to deal with $\mathcal{F}_s$-measurable parameters in the differential equations and to study eigenvalue-eigenvector problems for $\mathcal{D}_s$ with $\mathcal{F}_s$-measurable eigenvalues.

The key feature of weak time-derivatives in $[s, T]$ is the characterization of *conditional* (or *generalized*) *martingales*. By this terminology we mean processes u defined in the time interval $[s, T]$ with all the properties of martingales except for integrability, which is replaced by the weaker condition $\mathbb{E}_t[u_\tau] \in L^0(\mathcal{F}_t)$ for all $s \leqslant t \leqslant \tau \leqslant T$ (Shiryaev, 1996, Chapter VII, §1). Importantly, a process belongs to $\mathcal{U}_s^1$ and has null weak time-derivative in $[s, T]$ if and only if it is a conditional martingale (Proposition 12 in Appendix A). For a use of conditional martingales in portfolio optimization, see the recent Cerreia-Vioglio et al. (2022).

When the risk-neutral probability Q is considered, an example of conditional martingale is provided by the process of *futures prices* of a claim with expiry T at any t in $[s, T]$. On the contrary, under the forward measure, *forward prices* in $[s, T]$ for the settlement date T are conditional martingales (Musiela and Rutkowski, 2005, Sections 9.6 and 11.5). Both these processes exhibit null weak time-derivatives in $[s, T]$ with respect to different measures. The weak time-derivative in $[s, T]$ is also useful for the analysis of no-arbitrage prices because, after discounting by the money-market account, they are martingales under Q. Moreover, the weak time-derivative captures the drift of semimartingale processes and applies to a wide range of continuous-time pricing models. See Marinacci and Severino (2018) and the example in Subsection 3.1.

3 Pricing equation

We formulate and solve a *rate-adjusted* pricing equation for the valuation of random payments in a market with stochastic interest rates. We, then, compare the solution of the equation with the usual risk-neutral pricing formula. We interpret rate-adjusted prices as indifference prices and as conversion prices for hybrid securities. We finally study their properties in the long run and we describe the valuation of cashflows. Proofs can be found in Appendix B.

3.1 Rate-adjusted and no-arbitrage pricing

In the time interval $[s, T]$ we face the problem of evaluating an $\mathcal{F}_T$-measurable payoff h_T. As illustrated in Subsection 2.3, we can express the time t no-arbitrage price of h_T by using both the measure Q and the forward measure. Up to replacing the risk-neutral measure with F^T and instantaneous rates with bond yields, the right-hand side of eq. (4) formally traces no-arbitrage pricing with constant interest rates.

Generalizing Hansen and Scheinkman (2009), Marinacci and Severino (2018) show that, in an arbitrage-free market with constant interest rate r, asset prices satisfy the relation (1). In particular, the risk-neutral price process $\pi_t(h_T)$ is the unique solution in $\mathcal{U}_0^1$ of the no-arbitrage pricing equation

$$\begin{cases} \mathcal{D}_0 f_t = r \ f_t & t \in [0, T) \\ f_T = h_T \end{cases} \tag{7}$$

where $h_T \in L^1(\mathcal{F}_T, Q)$. This eigenvalue-eigenvector problem has its roots in the Perron-Frobenius theory and captures the essence of no-arbitrage. Indeed, it relates infinitesimal price increments of a possibly risky security with the interest rate deriving from a locally riskless investment. This approach is known in the literature since Cox and Ross (1976) derivation of Black-Scholes formula via a hedging portfolio. Moreover, the rate r, up to a sign change, defines the growth rate of the stochastic discount factor in the Hansen-Scheinkman decomposition, independently on the time horizon under consideration.

Example. We can consider a Black and Scholes (1973) market with a riskless bond and a risky security whose prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t dW_t^P$$

under the physical measure. Here, $r, \mu \in \mathbb{R}$, $\sigma > 0$ and W_t^P is a standard Wiener process. It is apparent that the bond price satisfies problem (7). As for the risky asset, under P the Wiener process W_t^P exhibits null weak time-derivative because it is a martingale, but the drift of X_t differs from rX_t. However, moving to the risk-neutral measure Q, the dynamics of X_t becomes

$$dX_t = rX_t dt + \sigma X_t dW_t^Q, \tag{8}$$

where W_t^Q is a Q-Wiener process, and so $\mathcal{D}_0 X_t = rX_t$. Hence, problem (7) is satisfied. In this arbitrage-free market, the risk-free and the risky security share the same drift coefficient under Q, given by the instantaneous rate.

The dynamics presented so far generalize to stochastic-rate settings by replacing r with the instantaneous rate Y_t. For example, Chapter 1 of Karatzas and Shreve (1998) derives similar dynamics to eq. (8) with random rates. The same approach is exploited by Heath et al. (1992) to get no-arbitrage restrictions on the forward rate drift. Nevertheless, there are two major flaws in these generalizations. First, the drift coefficient Y_t is unable to capture the stochastic discount factor growth rate over any time period, differently from the constant-rate case. Second, Y_t is a random process. Hence, it is not known ex ante (since it is floating over time) and it forbids an eigenvalue-eigenvector formulation of the problem in the spirit of Hansen and Scheinkman (2009).

Our generalization of problem (7) solves these issues for any given dynamics of interest rates. We employ the forward measure instead of Q and T-bond yields in place of short-term rates to formulate a suitable eigenvalue-eigenvector problem. Moreover, we provide a conditional version of the differential problem defined on any time window $[s, T]$ by using the weak time-derivative in $[s, T]$. The arising eigenvalue is an $\mathcal{F}_s$-measurable random variable (known at the beginning of the trading interval) and, consistently, the growth rate of the stochastic discount features the same $\mathcal{F}_s$-measurability.

To enter the details of our construction, we place ourselves in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, F^T)$, where we employ the forward measure. We solve in $\mathcal{U}_s^1$ the following pricing differential equation with random coefficient given by the yield to maturity $r_s^T \in L^0(\mathcal{F}_s)$, i.e.

$$\begin{cases} \mathcal{D}_s f_t = r_s^T \ f_t & t \in [s, T) \\ f_T = h_T \end{cases} \tag{9}$$

with $h_T \in L_s^1(\mathcal{F}_T, F^T)$. We refer to (9) as the *rate-adjusted pricing equation.*

Theorem 2 *There exists a unique solution of problem* (9) *in $\mathcal{U}_s^1$, given by*

$$\rho_t^T(s, h_T) = e^{-r_s^T (T-t)} \mathbb{E}_t^{F^T}[h_T] \qquad \forall t \in [s, T]. \tag{10}$$

We refer to $\rho_t^T(s, h_T)$ as the *rate-adjusted price* of h_T at time t in the interval $[s, T]$.

Theorem 2 is substantially an assessment of the conditional martingale property of the process $\{e^{r_s^T(T-t)}\rho_t^T\}_{t\in[s,T]}$ under F^T. This process is, in fact, a collection of forward prices for h_T. Under specific assumptions on asset and rate dynamics, the problem can be tackled from the perspective of the Feynman-Kaç partial differential equation, where null weak time-derivatives boil down to the zero-drift condition (see Subsection 6.2.1).

At any instant t in $[s, T]$, r_s^T is the only average rate employed by ρ_t^T for the valuation on h_T. The valuation instant is synchronous with the information set only for the initial ρ_s^T. In particular, ρ_s^T coincides with the no-arbitrage price π_s. The two coincide also at the terminal date.

In general, when $s < t < T$ the rate-adjusted price is different from the no-arbitrage price. Fixed any t, a bunch of valuations $\rho_t^T(s, h_T)$ are available, obtained by solving several problems as (9) defined on different time intervals $[s, T]$ with $s < t$. However, rate-adjusted prices with different starting points are consistent within them. Indeed, consider $s_1 \leqslant s_2 \leqslant t$ and $h_T \in L^1_{s_1}(\mathcal{F}_T, F^T)$. Then, the martingale property of forward prices ensures:

$$\mathbb{E}_{s_1}^{F^T}\left[e^{r_{s_2}^T(T-t)}\rho_t^T\left(s_2, h_T\right)\right] = \mathbb{E}_{s_1}^{F^T}\left[e^{r_{s_1}^T(T-t)}\rho_t^T\left(s_1, h_T\right)\right].$$

In addition, by moving s_2 to t from the left, if $r_{s_2}^T$ converges in probability to r_t^T, by the continuous mapping theorem we get

$$\rho_t^T\left(s_2, h_T\right) \xrightarrow{P} \pi_t\left(h_T\right), \qquad s_2 \to t^-.$$

The proper link between ρ^T and π is given by

$$\rho_t^T\left(s, h_T\right) = e^{-\left(r_s^T - r_t^T\right)(T-t)}\ \pi_t\left(h_T\right) = e^{r_s^T(t-s)}\frac{\pi_s\left(1_T\right)}{\pi_t\left(1_T\right)}\ \pi_t\left(h_T\right). \tag{11}$$

This equality can also be read as a parity relation between ρ^T and π:

$$e^{r_s^T(T-t)}\ \rho_t^T\left(s, h_T\right) = e^{r_t^T(T-t)}\ \pi_t\left(h_T\right),$$

where both sides deliver the price at time t of a forward contract on h_T for date T.

The difference between rate-adjusted and no-arbitrage prices is genuinely due to the term structure of interest rates in the market: in case rates are constant, the distortion between ρ^T and π disappears. In fact, problem (9) generalizes the dynamics of no-arbitrage prices of problem (7) for constant interest rates. ρ^T solves in $[s, T]$ the analogous differential equation of π, where interest rates are replaced by $\mathcal{F}_s$-measurable bond yields and the forward measure replaces the risk-neutral

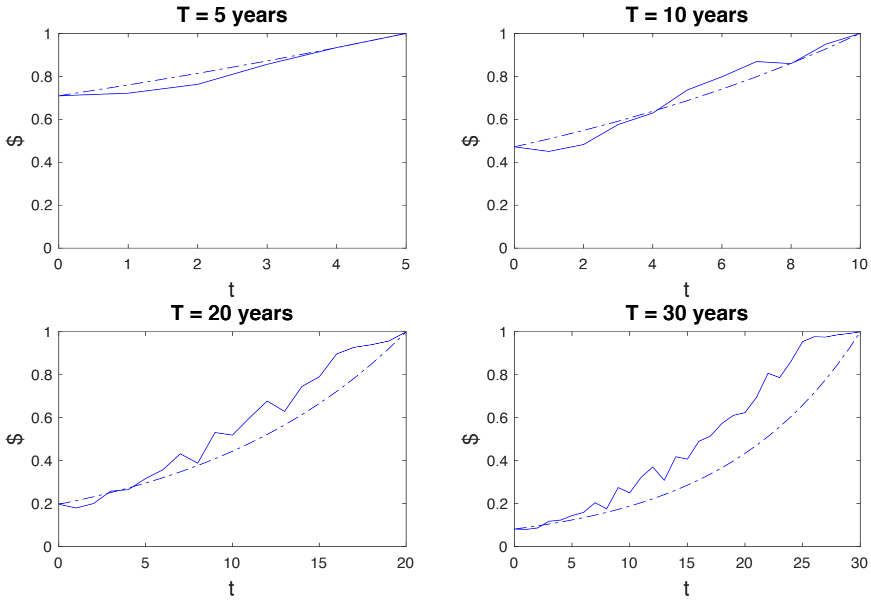


Figure 1: Fix $s = 0$ at January 1987 and consider increasing maturities T of U.S. Treasury bonds. Dashed lines depict the annual values of $\rho_t^T(0, 1_T)$ in $[0, T]$. Solid lines represent the (ex-post) realizations of no-arbitrage prices $\pi_t(1_T)$ at any year t until expiration.

one. In this perspective rate-adjusted prices may be seen as a generalization of no-arbitrage prices in floating-rate markets. Moreover, when ZCBs are considered, $\rho_t^T(s, 1_T)$ is exactly the no-arbitrage price of the ZCB in a parallel market in which interest rates are fixed and equal to the yield over $[s, T]$.

Figure 1 considers U.S. Treasury bonds in January 1987 with increasing expiry T of $5, 10, 20$ and 30 years. We fix $s = 0$ at January 1987 and consider annual t up to the redemption date. Data are provided by the Federal Reserve Board at daily frequency (Gürkaynak et al., 2007). In particular, the yield curve associated with U.S. Treasury bonds in January 1987 is increasing. In the four graphs of Figure 1 we plot the rate-adjusted prices $\rho_t^T(0, 1_T)$ of these securities and the ex-post realizations of no-arbitrage prices $\pi_t(1_T)$, observed in later years in the market. Treasury bonds rate-adjusted and no-arbitrage prices are indistinguishable at the initial date and near maturity. Different values of r_0^T may, however, induce an overestimation or underestimation of bond prices, which can be evaluated in future periods according to the actual realizations of bond prices in the market. Specifically, $\rho_t^T(0, 1_T) < \pi_t(1_T)$ if and only if $r_0^T > r_t^T$, a property that holds for any non-negative payoff h_T, too. The relation between yields at different times determines, indeed, the discrepancy between rate-adjusted and no-arbitrage valuation.

3.2 Rate-adjusted prices as indifference prices

Rate-adjusted prices can be interpreted as indifference prices that allow investors to hedge from interest rates variability. We bolster this intuition by describing a simple self-financing strategy in our market.

We consider an investor who incurs an expenditure of $\pi_s(1_T)$ at time s in order to buy a self-financing portfolio that delivers some units of a risky security with payoff h_T at maturity. Specifically, at date s she buys a T-ZCB. At a later time t she rebalances her allocation, worth $\pi_t(1_T)$, by purchasing the amount $\pi_t(1_T)/\pi_t(h_T)$ of the risky security, which is traded in the market at the no-arbitrage price. Then, she holds this portfolio until maturity.

We suppose that another investor faces the same market. She is less sophisticated than the previous one and pretends that the ZCB is traded at the fixed rate r_s^T. Her belief is consistent with the observation of the T-bond price at time s, that is $\pi_s(1_T) = e^{-r_s^T(T-s)}$, which coincides with $\rho_s^T(s, 1_T)$. Similarly to the first agent, she plans to go long on the ZCB at time s and to entirely liquidate her position at t in order to purchase all the units of the risky security that she can afford.

Since the second investor disregards the term structure of bond prices, she expects a position worth $e^{-r_s^T(T-t)}$ at time t and, consequently, an erroneous amount of the risky security at maturity. However, the two agents would end up with the same number of units of h_T if the second investor used ρ^T for pricing the risky security. Indeed, $\rho_t^T(s, h_T)$ is the theoretical price of h_T that makes the terminal values of both portfolios coincide.

This description substantiates the hedging nature of ρ^T. Although the first investor properly exploits the variability of rates, the second one pretends to face a flat term structure. Given an identical initial expenditure, they obtain the same outcome if the less sophisticated agent employs rate-adjusted prices for the valuation of marketed payoffs.

3.3 Rate-adjusted prices as conversion prices

We now provide an interpretation of rate-adjusted prices in terms of conversion prices of mandatory convertible bonds with contingent conversion prices. In line with the previous notation, we assume that the payoff h_T represents a stock price and we denote it by X_T.

We consider a convertible bond issued at time s with maturity T, unitary price and fixed rate r_s^T. At a given time $t < T$, the security is compulsorily converted into a proper amount of shares. The conversion price and ratio are determined at time t by no-arbitrage considerations.

Suppose that at time s the issuer of the hybrid security purchases $1/\pi_s(1_T)$ units of a T-ZCB. The total cashflow at s is null. Immediately before the conversion, the value of the position is

$\pi_t(1_T)/\pi_s(1_T)$ while the convertible bond is worth $e^{r_s^T(t-s)}$. On the one hand, the holdings of the issuer at time t correspond to $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$ shares with price $\pi_t(X_T)$. On the other, the bond conversion takes place according to the relation $e^{r_s^T(t-s)} = p_t q_t$, where p_t and q_t are $\mathcal{F}_t$-measurable conversion price and ratio. The absence of arbitrage opportunities implies that q_t needs to coincide with the amount of shares $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$. Otherwise, an initial null expenditure would ensure a positive outcome (in terms of stocks) at terminal date T. Since $p_t = e^{r_s^T(t-s)}/q_t$, we deduce that p_t takes the expression of eq. (11): p_t equals the rate-adjusted price $\rho_t^T(s, X_T)$.

The construction of the previous hybrid asset characterizes ρ^T as a conversion price from bonds to stocks in an arbitrage-free market. The reasoning highlights the ability of ρ^T to quantify the risk exposure when financing through fixed-rate securities in a market with stochastic rates.

3.4 Long-term relations between rate-adjusted and no-arbitrage prices

We now investigate the relation between ρ_t^T and π_t when the horizon T increases. Although the yields r_s^T and r_t^T converge to the same long-term yield, the asymptotic relations between ρ_t^T and π_t are delicate.

Proposition 3 *Under the Assumptions 1, if $h_T \in L_s^1(\mathcal{F}_T, F^T)$, then for any $t > s$*

$$\frac{\rho_t^T(s, h_T) - \pi_t(h_T)}{\mathbb{E}_t^{F^T}[h_T]} \xrightarrow{P} 0, \qquad T \to +\infty$$

and, in case h_T is also strictly positive,

$$\frac{\log \rho_t^T(s, h_T) - \log \pi_t(h_T)}{T - t} \xrightarrow{P} 0, \qquad T \to +\infty.$$

The asymptotic behaviors of Proposition 3 are valid for any choice of s and $t > s$, which are fixed before taking the limit across increasing horizons. However, convergences are taken after a proper rescaling. In the next proposition we focus on the ratio between rate-adjusted and no-arbitrage prices.

Proposition 4 *Under the Assumptions 1, if $h_T \in L_s^1(\mathcal{F}_T, F^T)$, then for all $t > s$*

$$\mathbb{E}_s^{F^T}\left[\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)}\right] \xrightarrow{P} e^{\left(r^\infty - r_s^t\right)(t-s)}, \qquad T \to +\infty$$

and

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty.$$

The difference between the long-term yield and the bond yield r_s^t determines the limit for the expected ratio between ρ_t^T and π_t under F^T. In addition, this ratio turns out to have a well-defined long-run limit in probability. Asymptotically ρ_t^T differs from π_t by a multiplicative $\mathcal{F}_t$-measurable factor associated with the discounted long bond. This factor is actually the transient component in the long-term pricing kernel decomposition of eq. (5).

3.5 Rate-adjusted prices as long-term prices

As described by Qin and Linetsky (2017) and recapped in eq. (5), under the Assumptions 1 the long-term growth rate of the pricing kernel $M_{s,t}$ is the long-term yield r^∞. Since r^∞ is the limit of r_s^T when T goes to infinity and r_s^T is the leading parameter of problem (9), we move T to infinity in this problem and analyze the solutions over increasing horizons. We begin with considering the *long-term* rate-adjusted problem

$$\begin{cases} \mathcal{D}_s f_t = r^\infty \ f_t & t \in [s, T) \\ f_T = h_T \end{cases} \tag{12}$$

with $h_T \in L_s^1(\mathcal{F}_T, F^\infty)$. Differently from problem (9), here the long-term yield replaces r_s^T and the long-term forward measure is employed. Theorem 2 ensures that problem (12) has a unique solution in $\mathcal{U}_s^1$, given by

$$\rho_t^\infty (s, h_T) = e^{-r^\infty (T-t)} \mathbb{E}_t^{F^\infty} [h_T] \qquad \forall t \in [s, T].$$

We refer to $\rho_t^\infty(s, h_T)$ as the *long-term rate-adjusted price* of h_T at time t in the interval $[s, T]$.

To investigate the convergence of ρ^T to ρ^∞ (when the horizon increases), we solve a sequence of differential problems related to a term structure of horizons, whose solutions tend to the solution of problem (12). To do so, we disentangle the instant at which the payoff under scrutiny is paid (τ) from the horizon ($T \geqslant \tau$). The rate-adjusted pricing problem of Subsection 3.1 can, then, be rewritten as

$$\begin{cases} \mathcal{D}_s f_t = r_s^T \ f_t & t \in [s, \tau) \\ f_\tau = h_\tau \end{cases} \tag{13}$$

with $h_\tau \in L_s^1(\mathcal{F}_\tau, F^T)$. If $\tau = T$, we retrieve problem (9). In the more general formulation considered here, the unique solution in $\mathcal{U}_s^1$ over $[s, \tau]$ is

$$\rho_t^T (s, h_\tau) = e^{-r_s^T (\tau-t)} \mathbb{E}_t^{F^T} [h_\tau] \qquad \forall t \in [s, \tau]. \tag{14}$$

This rate-adjusted price is fundamental for the valuation of cashflows in the next subsection. In addition, under mild assumptions, it converges to the long-term rate-adjusted price.

Proposition 5 *Under the Assumptions 1, suppose that $h_\tau \in L^1(\mathcal{F}_\tau, F^T)$ for all $T \geqslant \tau$ and $G^T_{t,\tau} h_\tau$ is convergent in $L^1(P)$ when T goes to infinity for all $t \in [s, \tau]$. Then, for all $t \in [s, \tau]$,*

$$\rho^T_t (s, h_\tau) \xrightarrow{P} \rho^\infty_t (s, h_\tau), \qquad T \to +\infty.$$

As a result, ρ^∞ can be properly interpreted as a rate-adjusted price for the long run. Indeed, the asset valuation through ρ^∞ exploits r^∞, which constitutes the stochastic discount factor long-term growth rate.

3.6 Valuation of cashflows

Eq. (14) describes the rate-adjusted price of a payoff h_τ in the time window $[s, T]$ with $s \leqslant \tau \leqslant T$. Disentangling the payment date τ from the horizon T permits to derive the rate-adjusted price for payoff streams. First, denote by $\alpha^T_t(s, h_\tau)$ the ratio between the rate-adjusted price and the no-arbitrage price of h_τ at any time $t \in [s, \tau]$:

$$\rho^T_t (s, h_\tau) = \alpha^T_t (s, h_\tau)\, \pi_t (h_\tau) = e^{-\left(r^T_s - r^\tau_t\right)(\tau - t)} \frac{\mathbb{E}^{F^T}_t [h_\tau]}{\mathbb{E}^{F^\tau}_t [h_\tau]} \pi_t (h_\tau). \tag{15}$$

Note that $\alpha^T_t (s, h_T)$ does not depend on h_T and it is consistent with eq. (11).

Consider now a cashflow $h = \{h_{\tau_i}\}^N_{i=1}$ with $N \in \mathbb{N}$, $s \leqslant \tau_1 < \cdots < \tau_N \leqslant T$ and $h_{\tau_i} \in L^1(\mathcal{F}_{\tau_i}, F^T)$ for all i. The rate-adjusted price $\rho^T_t(s, h)$ of the cashflow follows from standard linear pricing and the related rate adjustment is the ratio between $\rho^T_t(s, h)$ and the no-arbitrage price $\pi_t(h)$:

$$\rho^T_t (s, h) = \sum_{\tau_i \geqslant t} \rho^T_t (s, h_{\tau_i}) = \alpha^T_t (s, h)\, \pi_t (h), \tag{16}$$

$$\alpha^T_t (s, h) = \frac{\sum_{\tau_i \geqslant t} \alpha^T_t (s, h_{\tau_i})\, \pi_t (h_{\tau_i})}{\sum_{\tau_i \geqslant t} \pi_t (h_{\tau_i})}. \tag{17}$$

The evolution of α^T, as well as the difference between ρ^T and π, can be useful for quantifying the interest rate risk exposure of the cashflow over time. Indeed, the no-arbitrage price of h decomposes into the product of ρ^T and the inverse of α^T. Since the rate-adjusted price captures the long-term interest rate risk of h (as we further illustrate in Subsection 4.2), the short-term interest rate exposure flows into the adjustment. This approach is fruitful for the risk management of securities that feature long maturities and sensitivity to shocks in the term structure of rates as interest rate derivatives, life insurances and annuities (see Section 5). In addition, in a buy-and-hold portfolio strategy, the evolution of α^T can provide an indication of the interest rate risk to bear in case the position is unexpectedly closed before maturity.

4 Pricing kernel growth

From problem (9) it is clear that $\mathcal{D}_s\rho^T$ is weakly time-differentiable in $[s,T]$. The same holds for weak time-derivatives of higher orders. As a result, the rate-adjusted price ρ^T is infinitely weakly time-differentiable and so it belongs to $\mathcal{U}_s^\infty$. A parallel reasoning ensures the infinite weak time-differentiability of ρ^∞.

Since $\mathcal{D}_s(\xi_s u) = \xi_s \mathcal{D}_s u$ for all $u \in \mathcal{U}_s^\infty$ and $\xi_s \in L^0(\mathcal{F}_s)$, the weak time-derivative in $[s,T]$ defines an L^0-linear operator $\mathcal{D}_s : \mathcal{U}_s^\infty \to \mathcal{U}_s^\infty$ and ρ^T satisfies the eigenvalue-eigenvector problem

$$\mathcal{D}_s\rho^T = r_s^T \ \rho^T, \tag{18}$$

where the eigenvalue belongs to $L^0(\mathcal{F}_s)$. Accordingly, ρ^∞ solves

$$\mathcal{D}_s\rho^\infty = r^\infty \ \rho^\infty \tag{19}$$

where the eigenvalue is a positive number.

As in Hansen and Scheinkman (2009), we suppose that the payoff h_T is positive so that ρ^T and ρ^∞ are positive. Hence, ρ^T and ρ^∞ are *principal eigenvectors* related to r_s^T and r^∞ respectively. This property is key to the economic theory because it is in line with the Perron-Frobenius theory, usually employed in Markov environments and successfully applied by Ross (2015) and Qin and Linetsky (2016). Indeed, when rates are constant, the principal eigenvalue associated with a differential price operator turns out to be the growth rate of the stochastic discount factor (Hansen and Scheinkman, 2009).

In this section we analyze the relation between the eigenvalues of several differential pricing problems and pricing kernel growth rates in our stochastic-rate market. To build our theory, we introduce the notion of rate-adjusted pricing kernel. Proofs can be found in Appendix B.

4.1 Finite- and infinite-horizon pricing kernel decomposition

The long-term eigenvalue problem of eq. (19) actually exploits the pricing kernel long-term growth rate, that is the long-term yield. Indeed, as shown by Qin and Linetsky (2017), under the Assumptions 1 $M_{s,t}$ satisfies the long-run decomposition of eq. (5), i.e.

$$M_{s,t} = e^{-r^{\infty}(t-s)} \frac{b_s^{\infty}}{b_t^{\infty}} G_{s,t}^{\infty}.$$

The question, here, is whether an analogous property holds for the T-horizon problem of eq. (18). We need first to characterize the finite-horizon pricing kernel growth rate.

As introduced in Subsection 2.3, the pricing kernel $M_{s,t}$ satisfies

$$M_{s,t} = \frac{e^{-r_s^T(T-s)}}{e^{-r_t^T(T-t)}} G_{s,t}^T = \frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} \frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} G_{s,t}^T$$

for all $T > t + s$. In the next proposition we assess the asymptotic behaviors of the last three factors. Note that the ratio $\pi_s(1_T)/\pi_s(1_{T-t})$ is the return on a forward rate agreement between $T-t$ and T contracted at date s.

Proposition 6 *Under the Assumptions 1, for all $s > 0$ and $t > s$*

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} \xrightarrow{P} e^{-r^{\infty}(t-s)}, \qquad T \to +\infty$$

$$\frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} \xrightarrow{P} \frac{b_s^{\infty}}{b_t^{\infty}}, \qquad T \to +\infty$$

$$G_{s,t}^T \xrightarrow{P} G_{s,t}^{\infty}, \qquad T \to +\infty.$$

The first convergence shows that, in addition to being a limit of bond yields, r^{∞} is also a limit of continuously compounded forward rates contracted at s, as the convergence of $\pi_s(1_T)/\pi_s(1_{T-t})$ suggests. This approach is reminiscent of the constructions of Backus et al. (1989) and Alvarez and Jermann (2005). The second convergence involves the transient component of $M_{s,t}$. The limit of this factor is captured by the ratio of the discounted values of the long bond. Finally, the third convergence regards the permanent (or martingale) component of the pricing kernel. At any finite horizon, this term consists of the Radon-Nikodym density of the T-forward measure. When the horizon is infinite, the long-term forward measure appears.

Consistently with the three convergences of Proposition 6, when the horizon is finite we call growth term of $M_{s,t}$ the first ratio in the proposition, i.e.

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{-r_s^T(T-2s)+r_s^{T-t}(T-t-s)}. \tag{20}$$

Hence, $M_{s,t}$ features a random growth term in $L^0(\mathcal{F}_s)$ determined by bond yields at time s. In case interest rates are constant, this quantity actually reduces to $e^{-r(t-s)}$, which captures the growth of the pricing kernel $e^{-r(t-s)} L_{s,t}$.

As it is apparent from eq. (20), the growth rate of $M_{s,t}$ is not exactly r_s^T and so, differently from the long-term yield, it does not agree with the eigenvalue problem $\mathcal{D}_s \rho^T = r_s^T \rho^T$. Therefore, to isolate the growth rate r_s^T in finite horizons, we introduce the notion of *rate-adjusted pricing kernel*, which parallels the one of rate-adjusted prices.

4.2 Rate-adjusted pricing kernel

Given any payoff h_T, we can write no-arbitrage and rate-adjusted prices at time t as

$$\pi_\tau (h_T) = \mathbb{E}_\tau [M_{\tau,T} h_T], \qquad \rho_\tau^T (s, h_T) = \mathbb{E}_\tau \left[N_{\tau,T}^T h_T\right],$$

where, for all τ, t in $[s,T]$ with $\tau \leqslant t$,

$$M_{\tau,t} = \frac{e^{r_t^T (T-t)}}{e^{r_\tau^T (T-\tau)}} G_{\tau,t}^T, \qquad N_{\tau,t}^T = \frac{e^{r_s^T (T-t)}}{e^{r_s^T (T-\tau)}} G_{\tau,t}^T.$$

We call $N_{s,t}^T$ the *rate-adjusted pricing kernel*. In particular, we have $N_{s,t}^T = e^{\left(r_s^T - r_t^T\right)(T-t)} M_{s,t}$ and it coincides with $M_{s,t}$ when rates are constant. The adjustment coefficient between $N_{s,t}^T$ and $M_{s,t}$ is the inverse of the one between π and ρ^T pointed out in eq. (11). In addition, the expected values of $M_{s,t}$ and $N_{s,t}^T$ differ only in the yields r_s^t and r_s^T:

$$\mathbb{E}_s [M_{s,t}] = e^{-r_s^t (t-s)}, \qquad \mathbb{E}_s \left[N_{s,t}^T\right] = e^{-r_s^T (t-s)}.$$

Since $N_{s,t}^T = e^{-r_s^T (t-s)} G_{s,t}^T$, its growth rate is r_s^T, in agreement with the rate-adjusted eigenvalue problem $\mathcal{D}_s \rho^T = r_s^T \rho^T$. Moreover, differently from $M_{s,t}$, the rate-adjusted pricing kernel is explicitly dependent on the horizon T under scrutiny. In the next proposition we establish the convergence of $N_{s,t}^T$ when T becomes arbitrarily large.

Proposition 7 *Under the Assumptions 1, for all $s > 0$ and $t > s$,*

$$N_{s,t}^T \xrightarrow{P} e^{-r^\infty (t-s)} G_{s,t}^\infty, \qquad T \to +\infty.$$

Therefore, we define the *long-term rate-adjusted pricing kernel* by

$$N_{s,t}^\infty = e^{-r^\infty (t-s)} G_{s,t}^\infty \tag{21}$$

and r^∞ naturally arises as growth rate for $N_{s,t}^\infty$.

A formalization of the fact that r_s^T and r^∞ are the rate-adjusted pricing kernel growth rates for finite (or infinite) horizons can be easily obtained in differential terms. When the horizon is finite, we can consider the problem

$$\begin{cases} \mathcal{D}_s f_t = -r_s^T \ f_t & t \in [s, T] \\ f_s = 1 \end{cases} \tag{22}$$

under the physical measure. When the horizon is infinite, we analyze (still under P)

$$\begin{cases} \mathcal{D}_s f_t = -r^\infty \ f_t & t \in [s, +\infty) \\ f_s = 1. \end{cases} \tag{23}$$

The last problem uses weak time-derivatives in $[s, +\infty)$ that can be defined by replacing T with $+\infty$ in Definition 1. The module $\mathcal{U}_s$ modifies accordingly by omitting the left-continuity of processes u at T but additionally requiring that $\int_s^{+\infty} \mathbb{E}_s[|u_\tau|] d\tau$ belongs to $L^0(\mathcal{F}_s)$.

Theorem 8 *Under the measure P, $\{N_{s,t}^T\}_t$ solves problem* (22) *in $\mathcal{U}_s^1$ with finite T and $\{N_{s,t}^\infty\}_t$ solves problem* (23) *in $\mathcal{U}_s^1$ with infinite T.*

The last theorem formalizes the growth terms r_s^T and r^∞ for the rate-adjusted pricing kernel at finite and infinite horizons. The results are consistent with the eigenvalue problems $\mathcal{D}\rho^T = r_s^T \rho^T$ and $\mathcal{D}\rho^\infty = r^\infty \rho^\infty$ satisfied by rate-adjusted prices.

Problems (22) and (23) generalize the differential relation $\mathcal{D}_s M = -rM$ which is satisfied by the pricing kernel when interest rates are constantly equal to r. When rates are stochastic, in standard diffusive models, the pricing kernel follows the dynamics $dM_{s,t} = -Y_t M_{s,t} dt - \nu_t M_{s,t} dW_t^P$, where ν_t is the market price of risk (see Subsection 6.2.2). The stochastic rate is present in the drift of the pricing kernel. However, any Y_t alone is not able to capture the growth rate of $M_{s,t}$ and it is not known ex ante. This last feature forbids any eigenvalue-eigenvector formulation of the problem of identifying a growth rate for $M_{s,t}$.

Problems (22) and (23) are not satisfied by the pricing kernel $M_{s,t}$ when interest rates are stochastic as well as no-arbitrage prices do not solve problems (9) and (12). The deep reason for these failures can be understood through the lens of the Hansen-Scheinkman decomposition.

By comparing eq. (21) with the long-term decomposition of the pricing kernel in eq. (5), it is apparent that $N_{s,t}^\infty$ features the same growth rate r^∞ of $M_{s,t}$, as well as the same martingale component $G_{s,t}^\infty$. Nevertheless, the transitory component of $N_{s,t}^\infty$ is deterministic and equal to 1. Therefore, employing rate-adjusted prices for asset valuation means using a stochastic discount factor that is free from transitory effects in its long-term Hansen-Scheinkman decomposition. See also the previous eq. (6) that highlights the role of the transient component of $M_{s,t}$ in the price of any marketed payoff.

When interest rates are constant, the transient component of the stochastic discount factor is 1 and so the difference between $M_{s,t}$ and $N^{\infty}_{s,t}$ vanishes. On the contrary, when rates are stochastic, a trivial temporary term can be retrieved only in $N^{\infty}_{s,t}$. For this reason, rate-adjusted prices allow us to generalize both price and stochastic discount factor dynamics from the constant-rate case. Rate-adjusted prices aggregate infinitesimal randomness to long-run risk exposure because $N^{T}_{s,t}$ is free from any transitory component. Moreover, bond yields are the proper financial variables that translate local riskiness to long-run risks through rate-adjusted prices. Table 1 summarizes our findings.

Table 1: **Summary of results**. Comparison of numéraires, pricing relations and pricing kernel growth rates between constant and stochastic rates. Finite and infinite horizons are considered. In the stochastic case, rate-adjusted prices and pricing kernels are considered.

	Constant rate	Stochastic rates	
		finite T	infinite T
Numéraire	money market	T-ZCB	long bond
Measure	Q	F^T	F^∞
Yield	r	r^T_s	r^∞
Price	π	ρ^T	ρ^∞
Return-rate relation	$\mathcal{D}_0\pi = r\pi$	$\mathcal{D}_s\rho^T = r^T_s\rho^T$	$\mathcal{D}_s\rho^\infty = r^\infty\rho^\infty$
Pricing kernel growth rate	r	r^T_s	r^∞

5 Applications of rate-adjusted pricing

The introduction of rate adjustments is fruitful in several areas of finance to disentangle the long- and short-term interest risk of cashflows.

5.1 Interest rate swaps

Consider an interest rate swap contract with periodic payment exchanges at dates $\tau_1, \ldots, \tau_N$ in the time interval $[0, T]$ with $\tau_N = T$. The reference period for the first payment is $[\tau_0, \tau_1]$ with $0 < \tau_0 < \tau_1$. At any payment date τ_i, two parties exchange a fixed leg K with a floating leg made by the LIBOR rate $\mathcal{L}(\tau_{i-1}, \tau_i)$, which satisfies $(1 + (\tau_i - \tau_{i-1})\mathcal{L}(\tau_{i-1}, \tau_i))\pi_{\tau_{i-1}}(1_\tau) = 1$. Hence, the cashflow h of the swap payer is given by $h_{\tau_i} = (\mathcal{L}(\tau_{i-1}, \tau_i) - K)(\tau_i - \tau_{i-1})$ for any $i = 1, \cdots, N$.

Then, the no-arbitrage price of any h_{τ_i} at time $t \in [0, \tau_i]$ is

$$\pi_t\left(h_{\tau_i}\right) = \begin{cases} \pi_t\left(1_{\tau_{i-1}}\right) - \pi_t\left(1_{\tau_i}\right) - k\left(\tau_i - \tau_{i-1}\right)\pi_t\left(1_{\tau_i}\right) & t \in [0, \tau_{i-1}) \\ h_{\tau_i}\pi_t\left(1_{\tau_i}\right) & t \in [\tau_{i-1}, \tau_i] \end{cases}$$

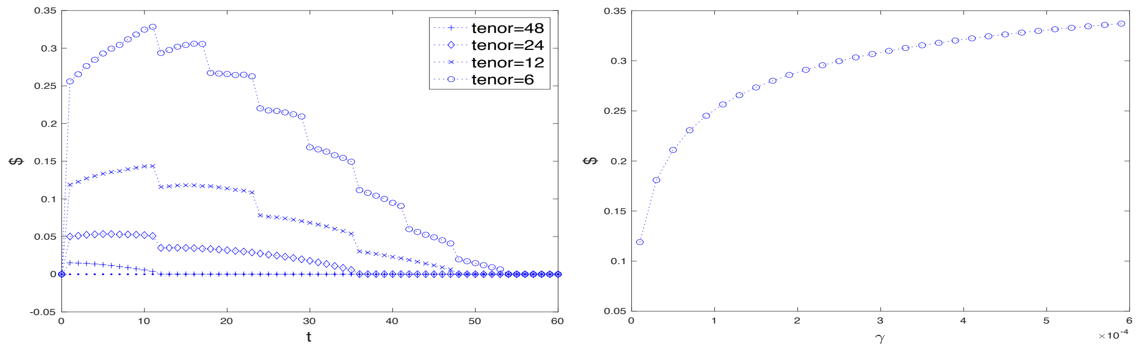
and the premium K is set in a way that $\pi_0\left(h\right) = 0$. We compute the no-arbitrage and the rate-adjusted price of the swap following Subsection 3.6.

We do a Monte Carlo simulation by assuming a Vasicek (1977) term structure of rates (see Subsection 6.1.1). We fix the first payment date τ_1 and the maturity T but we increase the number of exchanges in between. In so doing, we show that the rate-adjusted price relies mainly on the *maturity* of the swap, while the rate adjustment depends on the *tenor*. Indeed, short tenors are associated with a large short-term interest rate risk, which is much less relevant when payment exchanges are infrequent.

The main steps of the simulation are the following. From a random sample we simulate several Wiener processes under the physical measure. We then simulate the Vasicek interest rate process, as well as ZCB prices, yields to maturity and the Radon-Nikodym derivatives L_T and G_T^T. After that, we determine K, the no-arbitrage prices $\pi_t(h_{\tau_i})$ of each swap payment at any t in $[0, T]$ and the no-arbitrage price $\pi_t(h)$ of the swap contract. By eq. (15), we compute $\alpha_t^T(0, h_\tau)$ for any swap payment and, from eq. (17), we obtain $\alpha_t^T(0, h)$ for the whole swap. We finally compute the rate-adjusted price $\rho_t^T(0, h)$ of the swap from eq. (16), take the difference between $\rho_t^T(0, h)$ and $\pi_t(h)$ at any t and average across scenarios. We repeat the algorithm for different tenors.

We plot in the left panel of Figure 2 the difference between $\rho^T(0, h)$ and $\pi(h)$ of the swap contracts for decreasing tenors of 48, 24, 12 and 6 months, after setting $\tau_0 = 12$ and $\tau_N = 60$ months. Discounting with a constant rate r_0^T makes a relatively small discrepancy from no-arbitrage pricing when a single payment at T is considered. However, the difference between the no-arbitrage and the rate-adjusted price of the swap increases when the tenor shortens (and the number of payments increases): short tenors are sensitive to temporary interest rate variations and this sensitivity is captured by the rate adjustment.

The analysis proposed here becomes more interesting when dealing with a family of interest rate derivatives with irregular cashflows, where the exposure to short- or long-term interest risk is unclear. For each of them, the rate-adjusted price can be still computed from the no-arbitrage price and the difference between the two prices permits to quantify the assets exposure to short-term interest rate risk and to put them in order according to that.



(a) Interest rate swaps.

(b) Whole life insurance.

Figure 2: Left panel: difference between $\rho_t^T(0,h)$ and $\pi_t(h)$ of the interest rate swaps of Subsection 5.1 over time. We set $\tau_0 = 12$ and $\tau_N = 60$ months. Decreasing tenors of 48, 24, 12 and 6 months are considered. Right panel: difference between $\pi_0(h)$ and $\rho_0^\infty(0,h)$ of the whole life insurance of Subsection 5.2. The horizontal axis contains the parameter γ, which increases with the insured's age at subscription. In both panels a Vasicek term structure of rates is assumed on monthly basis.

5.2 Whole life insurance actuarial present value

Consider a whole life insurance that pays a benefit equal to 1 when the subscriber dies. The contract has no preset horizon and the date of the unique payment is unknown ex ante. This date is related to the time-until-death, a random variable with probability density function $\wp$ on $[0,+\infty)$, that represents the difference between the insured's age at death and her age at the subscription. Subscribers with different ages feature different future lifetime, and so different $\wp$. See Chapters 3 and 4 in Bower et al. (1997).

The actuarial present value of the insurance is the expectation of the present value of the payment. This value summarizes the overall exposure of the insurance company to the obligation of the contract. Since the horizon of the insurance policy is potentially very large, the actuarial present value depends on the forecast term structure of rates. See, e.g., Panjer and Bellhouse (1980) for the use of stochastic rates in the valuation of life contingencies. The introduction of rate adjustments additionally permits to disentangle the short- and long-term interest rate risk beared by the insurance company.

Suppose that the physical measure (used by the insurance company) coincides with the risk-neutral measure. In this case, in line with the previous notation, we can denote the actuarial present value by $\pi_0(h)$ and consider the *rate-adjusted* actuarial present value $\rho_0^\infty(0,h)$. Here h represents the unitary random payment of the whole life insurance (the subscriber's age is given):

$$\pi_0(h) = \int_0^{+\infty} e^{-r_0^\tau \tau} \wp(\tau) d\tau, \qquad \rho_0^\infty(0,h) = \int_0^{+\infty} e^{-r^\infty \tau} \wp(\tau) d\tau.$$

We assume that the time-until-death has cumulative distribution function $\mathcal{P}(\tau) = 1 - e^{-\gamma\tau^3}$ and in the simulations we move the parameter γ from 0.00001 to 0.00059. The distribution is unimodal and increasing γ means moving the peak of time-until-death towards the date of the insurance subscription. This means considering shorter future lifetimes or, equivalently, increasing the subscriber's age. The used monthly time grid involves 900 months after the subscription for all γ. Interest rates are simulated following Vasicek (1977) with the specifications of Subsection 6.1.1.

The right panel of Figure 2 represents the difference between π_0 and ρ_0^∞ when γ increases. The difference is small when γ is low and becomes sizable for high values of the parameter. The rate-adjusted actuarial present value differs more from the actuarial present value when old subscribers (high γ) are considered. Their future lifetime is short and so short-term rates have an important role in the computation of their actuarial present value. This valuation contrasts with the rate-adjusted one, which responds substantially to long-term interest rate shocks. The difference between the two actuarial values captures the short-term interest rate risk. On the contrary, ρ_0^∞ is more similar to π_0 for young insured (low γ) because their time-until-death is large and so their actuarial present value exploits heavily the long-run rates. Hence, considering the rate adjustment provides insights on the short and long-term interest rate risk exposure of the insurance company designing the contract.

5.3 Mortgage renegotiation

Consider a fixed-rate mortgage that features periodic payments equal to c at dates $\tau_1, \dots, \tau_N$ in the time interval $[0, T]$ with $\tau_N = T$. In line with our notation, we denote by h the stream of payments c in the annuity. The amount of money demanded by the borrower is v_0 and the bank imposes the rate ξ, so that $v_0 = \sum_{i=1}^N e^{-\xi\tau_i} c$.

At time $t > 0$ the borrower wants to assess whether a renegotiation of the loan is convenient. According to the term structure of rates at t, the market value of the remaining sequence of payments is $\pi_t(h) = \sum_{\tau_i > t}^T e^{-r_t^{\tau_i}(\tau_i - t)} c$ while the present value obtained from the fixed-rate discounting is $v_t = \sum_{\tau_i > t}^T e^{-\xi(\tau_i - t)} c$. The borrower is willing to ask for a renegotiation if $v_t > \pi_t$.

If $\xi = r_0^T$, then v_t turns out to be the rate-adjusted price of the remaining payments in the annuity, i.e. $v_t = \rho_t^T(0, h)$, and the renegotiation is profitable when the rate-adjusted price exceeds the no-arbitrage price $\pi_t(h)$. This requirement boils down to a condition for the rate adjustment of eq. (17), namely $\alpha_t^T(0, h) > 1$, where $\alpha_t^T(0, h)$ equals $\sum_{\tau_i > t}^T e^{-r_0^T(\tau_i - t)} / \sum_{\tau_i > t}^T e^{-r_t^{\tau_i}(\tau_i - t)}$. In general, if ξ is arbitrary, the latter ratio rewrites with ξ instead of r_0^T.

In a nutshell, the mortgage renegotiation rests on the relation between the current term structure of rates and the preset fixed rate. If the latter is equal to r_0^T, the trade-off between floating and fixed rates is captured exactly by the adjustment between no-arbitrage and rate-adjusted prices.

5.4 Pension fund withdrawal

Consider a worker that contributes to a pension plan by regularly devolving a given amount since date s. The fund guarantees a constant interest rate equal to ξ_s. Under some circumstances the worker is allowed to withdraw the total amount of her (capitalized) payments. She considers this possibility at time $t > s$ when her contribution amounts to q_t. The value of q_t that the fund would guarantee at T is $e^{\xi_s(T-t)}q_t$. On the contrary, according to the current yield curve, the future value of q_t at T would be $q_T = e^{r_t^T(T-t)}q_t$. As a result, the withdrawal is profitable when $e^{r_t^T(T-t)} > e^{\xi_s(T-t)}$. If $\xi_s = r_s^T$, this condition translates into a rate adjustment $\alpha_t^T(s, q_T)$ exceeding 1. Indeed, the comparison of the future values of q_t rests on the different discounting related to constant- and stochastic-rate valuations. The rate adjustment captures the trade-off between the two discounting ways when the constant rate is r_s^T.

6 Single-factor affine interest rate models

We provide an illustration of our theory in arbitrage-free markets with diffusive short-term rates. In particular, we devote some attention to the Feynman-Kaç partial differential equations satisfied by rate-adjusted and no-arbitrage prices in Subsection 6.2.1. Subsection 6.1 considers a fixed-income market while Subsection 6.2 involves a market with both stocks and ZCBs.

6.1 Pricing in a fixed-income market

We start comparing the drifts of ZCB prices under the measures P, Q and F^T in a fixed-income market. Then, we make a comparison with rate-adjusted prices. When the processes under scrutiny are weakly time-differentiable, the drift is the weak time-derivative (Marinacci and Severino, 2018) and Theorem 2 applies. We use Itô's formula extensively, implicitly postulating that the processes under consideration are continuously differentiable with respect to time and twice continuously differentiable with respect to the spatial variable.

We assume that instantaneous rates follow the diffusion process $dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t^P$ in the time window $[s, T]$. Here, μ and σ are measurable functions of t and Y_t, and W_t^P denotes a Wiener process under the physical measure. The T-ZCB price at time t is function of t and Y_t and

Itô's formula permits to determine its drift and diffusion coefficient:

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \tilde{\mu}(t, Y_t)\, dt + \tilde{\sigma}(t, Y_t)\, dW_t^P.$$

We indicate by ν the market price of risk process $\nu_t = (\tilde{\mu}(t, Y_t) - Y_t)/\tilde{\sigma}(t, Y_t)$. By Girsanov's theorem, we build a Wiener process under the risk-neutral measure Q through the stochastic differential $dW_t^Q = dW_t^P + \nu_t dt$ and so

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = Y_t\ dt + \tilde{\sigma}(t, Y_t)\, dW_t^Q.$$

Under Q the drift coefficient of no-arbitrage bond prices is the instantaneous rate. When considering the forward measure, the dynamics of $\pi_t(1_T)$ become

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \left(Y_t + \tilde{\sigma}^2(t, Y_t)\right) dt + \tilde{\sigma}(t, Y_t)\, dW_t^{F^T}, \tag{24}$$

where $W_t^{F^T}$ is a Wiener process under F^T satisfying $dW_t^{F^T} = dW_t^Q - \tilde{\sigma}(t, Y_t)dt$. See Chapter 3 of Brigo and Mercurio (2006). Differently from the Q-dynamics, the drift under F^T depends on the diffusion coefficient. This feature is not present in T-bond rate-adjusted prices because their differential satisfies

$$\frac{d\rho_t^T(s, 1_T)}{\rho_t^T(s, 1_T)} = r_s^T\ dt \tag{25}$$

for any measure under consideration. In agreement with Theorem 2, the weak time-derivative in $[s, T]$ of $\rho_t^T(s, 1_T)$ is $r_s^T \rho_t^T(s, 1_T)$, while under F^T the candidate weak time-derivative of $\pi_t(1_T)$ is $(Y_t + \tilde{\sigma}^2(t, Y_t))\pi_t(1_T)$. Since $Y_t + \tilde{\sigma}^2(t, Y_t)$ is floating over time, under F^T it is unlikely to find an eigenvalue-eigenvector formulation as problem (9) by using no-arbitrage prices instead of rate-adjusted prices when interest rates are stochastic.

6.1.1 Zero-coupon bonds and call options in Vasicek (1977)

In single-factor affine term structure models – Vasicek (1977), Cox, Ingersoll, and Ross (1985) and others – the price of a T-ZCB depends on the instantaneous rate Y_t through the exponential relation $\pi_t(1_T) = e^{A(t,T)-B(t,T)Y_t}$, with A and B deterministic functions. By Itô's differential rule, the diffusion coefficient of the no-arbitrage price $\pi_t(1_T)$ is $\tilde{\sigma}(t, Y_t) = -B(t, T)\sigma(t, Y_t)$. However, the dynamics of the related rate-adjusted price is still the one of eq. (25).

As an example, we consider the Vasicek (1977) model, in which the coefficients of the instantaneous rate are $\mu(t, Y_t) = k\theta - (k - \sigma\xi)Y_t$ and $\sigma(t, Y_t) = \sigma$, where $k, \theta, \sigma, \xi > 0$ and the market price of risk is ξY_t. Thus, the evolution of Y_t under Q is given by $dY_t = k(\theta - Y_t)\,dt + \sigma dW_t^Q$. The short-term rate is mean-reverting towards the value θ in the long run at a speed dictated by k. In addition, volatility is constant over time. Under this specification,

$$A(t,T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4k}B^2(t,T)$$

and $B(t,T) = (1 - e^{-k(T-t)})/k$. See Section 3.2 of Brigo and Mercurio (2006). Therefore, the drift coefficient of the no-arbitrage bond price under F^T is $Y_t + \left(1 - e^{-k(T-t)}\right)^2 \sigma^2/k^2$. When the horizon T becomes infinitely large, this term converges a.s. to $Y_t + \sigma^2/k^2$. On the contrary, the drift parameter r_s^T of the rate-adjusted price converges a.s. to the long-term yield $r^\infty = \theta - \sigma^2/2k^2$.

To provide an example on derivative pricing, we consider a European call option with expiration τ and strike price c over a T-ZCB with $T > \tau$. When short-term rates follow Vasicek dynamics, the no-arbitrage price of the option at any time t between 0 and τ is $\pi_t(h_\tau) = \pi_t(1_T)\mathcal{N}(q) - c\pi_t(1_\tau)\mathcal{N}(q - \hat{\sigma})$, where $\mathcal{N}$ denotes the cumulative distribution function of a standard Gaussian,

$$q = \frac{\log(\pi_t(1_T)) - \log(c\pi_t(1_\tau)) + \hat{\sigma}^2/2}{\hat{\sigma}},$$

$$\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2(k-\sigma\xi)(\tau-t)}}{2(k-\sigma\xi)} \frac{\left(1 - e^{-(k-\sigma\xi)(T-\tau)}\right)^2}{(k-\sigma\xi)^2}.$$

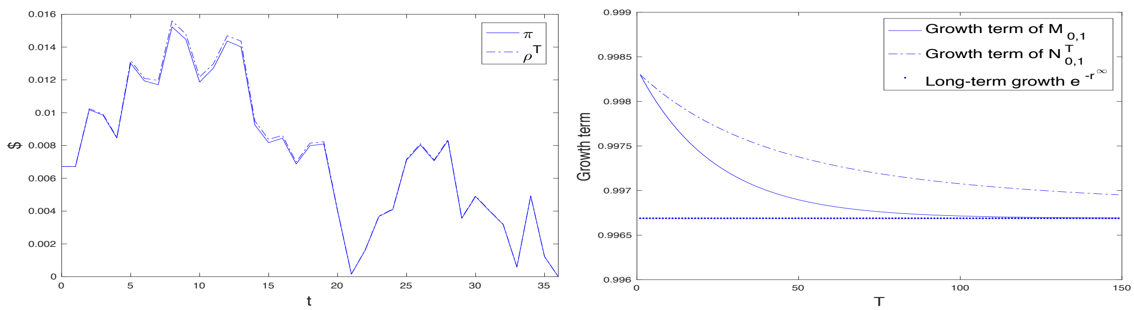
See Jamshidian (1989). The rate-adjusted price of the option can be easily obtained from the no-arbitrage price through eq. (11). We plot both price processes in the left panel of Figure 3, where we use the parameters $k = 0.5$, $\theta = 0.04$, $\sigma = 0.01$, $\xi = 0.2$ and $Y_0 = 0.02$ on a monthly time grid and we set $T = 72$ months, $\tau = 36$ months and $c = 0.89$. The difference between the two quantities may be appreciated mainly at some intermediate months.

6.1.2 Long-term relations for any payoff and pricing kernel growth

We now consider a generic attainable payoff $h_T \in L_s^1(\mathcal{F}_T, F^T)$ in a market with exponential affine interest rates. The ratio between the rate-adjusted price and the no-arbitrage price depends on the instantaneous rates at instants s and t:

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} = e^{\{A(s,T) - B(s,T)Y_s\}\frac{T-t}{T-s} - A(t,T) + B(t,T)Y_t}.$$

In addition, the long-run relation between ρ^T and π of Proposition 4 can be determined explicitly



(a) Call option on a ZCB.

(b) Growth terms.

Figure 3: Left panel: realizations of $\rho_t^\tau(0,h_\tau)$ and $\pi_t(h_\tau)$ for a European call option on a ZCB as described in Subsection 6.1.1. No-arbitrage prices are represented by solid lines and rate-adjusted prices by dashed lines. Right panel: term structure of growth terms of the pricing kernel $M_{0,1}$ and the rated-adjusted pricing kernel $N_{0,1}^T$ in Vasicek model. The solid line represents the growth term of $M_{0,1}$ for increasing horizons T, while the dashed line regards the growth term of $N_{0,1}^T$. The horizontal line is the long-term growth e^{-r^∞}.

under Vasicek assumptions:

$$\frac{\rho_t^T\left(s,h_T\right)}{\pi_t\left(h_T\right)} \xrightarrow{a.s.} e^{-\frac{1}{k}(Y_s-Y_t)}, \qquad T\to+\infty.$$

This limit is determined by the speed parameter k. If k is high, the two prices are almost indistinguishable for large maturities.

We finally focus on the pricing kernel between s and t in case bond yields are affine. The growth term of $M_{s,t}$ is an exponential function of Y_s, namely

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s\left(1_{T-t}\right)} = e^{A(s,T)\left(\frac{T-2s}{T-s}\right)-A(s,T-t)-\left\{B(s,T)\left(\frac{T-2s}{T-s}\right)-B(s,T-t)\right\}Y_s}.$$

When the terminal date T goes to infinity, this term converges a.s. to $e^{-r^\infty(t-s)}$, as expected. In the right panel of Figure 3 we display this convergence in a Vasicek model with $s=0$, $t=1$ and the parameters of Subsection 6.1.1. In addition, we graphically compare the growth term of $M_{0,1}$ with the one of the rate-adjusted pricing kernel $N_{0,1}^T$. The two terms share the same long-run limit.

6.2 Pricing with stocks and bonds

The situation is more involved when the market is generated by a set of risky assets with prices $X_t^{(1)},\dots X_t^{(N)}$, beyond ZCBs. Although all the drift coefficients of no-arbitrage prices of such securities coincide with Y_t under Q, the drifts of these prices under F^T are additionally affected by the correlation between the instantaneous rate and the idiosyncratic random component of the asset under consideration (see, e.g. Rabinovitch, 1989). On the contrary, the drift coefficient of the rate-adjusted price of any security is always equal to the yield r_s^T. To further elucidate the issue, we borrow the dynamics of rates and stock prices from Appendix B of Brigo and Mercurio (2006).

We assume that short-term rates move as in Vasicek model in the time interval $[s, T]$, with the same dynamics as Subsection 6.1.1. Then, we consider a stock price X_t that follows a geometric Brownian motion with volatility $\eta > 0$, correlated with interest rates shocks. The instantaneous correlation parameter between the two underlying Wiener processes is ϕ. We can make the two sources of randomness orthogonal and consider, without loss of generality,

$$\begin{cases} dX_t = X_t Y_t \; dt + \eta X_t \left(\phi dW_t^Q + \sqrt{1-\phi^2} dZ_t^Q \right) \\ dY_t = k\left(\theta - Y_t\right) dt + \sigma dW_t^Q, \end{cases}$$

where W_t^Q and Z_t^Q are independent Wiener processes. Under F^T, we get

$$\begin{cases} dX_t = X_t \left(Y_t - \frac{\phi\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right) dt + \eta X_t \left(\phi dW_t^{F^T} + \sqrt{1-\phi^2} dZ_t^{F^T} \right) \\ dY_t = \left(k\left(\theta - Y_t\right) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right) dt + \sigma dW_t^{F^T}. \end{cases}$$

It is now apparent that the correlation parameter ϕ impacts on the drift of X_t under the forward measure.

In the following subsections we derive the Feynman-Kaç PDEs satisfied by no-arbitrage and rate-adjusted prices in this market and the dynamics of the related pricing kernels.

6.2.1 Feynman-Kaç partial differential equations

Consider a contingent claim $h_T \in L_s^1(\mathcal{F}_T, F^T)$ which is a continuous function of X_T and Y_T. We determine the Feynman-Kaç PDEs necessarily satisfied by the two prices. We assume that $\pi_t(h_T)$ and $\rho_t^T(s, h_T)$ are continuously differentiable with respect to t and twice continuously differentiable with respect to the variables $X_t = x$ and $Y_t = y$. For a comprehensive treatment of the topic, see Karatzas and Shreve (1991), Section 4.4. The detailed derivations of the following PDEs are available upon request.

The discounted price $e^{-\int_s^t Y_\tau d\tau}\pi_t(h_T)$ is a Q-martingale. Therefore, by setting its drift equal to zero under Q, we get the Feynman-Kaç PDE for π:

$$\frac{\partial \pi}{\partial t} + xy\frac{\partial \pi}{\partial x} + k\left(\theta - y\right)\frac{\partial \pi}{\partial y} + \frac{\eta^2 x^2}{2}\frac{\partial^2 \pi}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2 \pi}{\partial y^2} + \phi\sigma\eta x \frac{\partial^2 \pi}{\partial x \partial y} = y\pi \tag{26}$$

with terminal condition $\pi_T(h_T) = h_T$. In case the correlation parameter ϕ is null and interest rates are constant, the usual Black-Scholes PDE arises.

Regarding the rate-adjusted price, since the forward price $e^{r_s^T(T-t)}\rho_t^T(s,h_T)$ is an F^T-martingale, we set its drift equal to zero under F^T. Then, the Feynman-Kaç PDE for ρ^T is

$$\frac{\partial\rho^T}{\partial t}+x\left(y-\frac{\phi\sigma\eta}{k}\left(1-e^{-k(T-t)}\right)\right)\frac{\partial\rho^T}{\partial x}+\frac{\eta^2x^2}{2}\frac{\partial^2\rho^T}{\partial x^2}+\frac{\sigma^2}{2}\frac{\partial^2\rho^T}{\partial y^2}$$
$$+\left(k(\theta-y)-\frac{\sigma^2}{k}\left(1-e^{-k(T-t)}\right)\right)\frac{\partial\rho^T}{\partial y}+\phi\sigma\eta x\frac{\partial^2\rho^T}{\partial x\partial y}=r_s^T\rho^T \tag{27}$$

with $\rho_T^T=h_T$.

Observe the right-hand sides in eq. (26) and (27). They contain the instantaneous rate y for π and the yield to maturity r_s^T for ρ^T. The coefficients of spatial first-order derivatives are also different but the dissimilarity reduces when the speed k increases. In addition, the coefficients of $\partial\pi/\partial x$ and $\partial\rho^T/\partial x$ coincide when the correlation parameter is null. Hence, the disparity between π and ρ^T can also be seized through the solution of different parabolic PDEs.

6.2.2 Pricing kernel dynamics

We now explicitly establish the evolution of $M_{s,t}$ and $N_{s,t}^T$ in our market. Beyond the money market account $\{e^{\int_s^t Y_\tau d\tau}\}_{t\in[s,T]}$, the prices of the assets that generate the market satisfy

$$\begin{cases} dX_t=X_tY_t\,dt+\eta X_t\left(\phi dW_t^Q+\sqrt{1-\phi^2}dZ_t^Q\right)\\ d\pi_t(1_T)=\pi_t(1_T)Y_tdt-\pi_t(1_T)B(t,T)\sigma dW_t^Q,\end{cases}$$

where the dynamics of $\pi_t(1_T)$ are derived in Subsection 6.1.1. At the same time, under the physical measure,

$$\begin{cases} dX_t=X_t\mu_t^Xdt+\eta X_t\left(\phi dW_t^P+\sqrt{1-\phi^2}dZ_t^P\right)\\ d\pi_t(1_T)=\pi_t(1_T)\mu_t^Pdt-\pi_t(1_T)B(t,T)\sigma dW_t^P,\end{cases}$$

where μ_t^X and μ_t^P are adapted processes. They are related to the drifts under Q via the bivariate process of market price of risk $[\nu_t^W,\nu_t^Z]'$ such that

$$\left[dW_t^Q,\ dZ_t^Q\right]'=\left[\nu_t^W,\ \nu_t^Z\right]'dt+\left[dW_t^P,\ dZ_t^P\right]'.$$

By assuming that $\mu_t^P=(1-\xi B(t,T)\sigma)Y_t$ for some $\xi>0$, we obtain

$$\nu_t^W=\xi Y_t,\qquad\qquad \nu_t^Z=\frac{\mu_t^X-Y_t-\eta\phi\nu_t^W}{\eta\sqrt{1-\phi^2}},$$

where ν_t^W is in line with the usual approach to Vasicek short-term rates.

Now consider the pricing kernel $M_{s,t}$. Since the processes defined by $M_{s,t}H_t$, where H_t is each of X_t, $\pi_t(1_T)$ and $e^{\int_s^t Y_\tau d\tau}$, are (conditional) P-martingales in $[s,T]$, their drifts are null and the dynamics of $M_{s,t}$ turns out to be

$$dM_{s,t} = -Y_t M_{s,t}dt - \nu_t^W M_{s,t}dW_t^P - \nu_t^Z M_{s,t}dZ_t^P.$$

The differential of the rate-adjusted pricing kernel $N_{s,t}^T$ can be inferred from the multiplicative relation $N_{s,t}^T = \pi_t(1_T)e^{r_s^T(T-t)}M_{s,t}$ by applying Itô's product rule. As a result, we have

$$dN_{s,t}^T = -r_s^T N_{s,t}^T dt - (\nu_t^W + B(t,T)\sigma)N_{s,t}^T dW_t^P - \nu_t^Z N_{s,t}^T dZ_t^P.$$

As expected, the dynamics of $N_{s,t}^T$ coincide with the ones of $M_{s,t}$ when interest rates are constant. Indeed, σ is null and the yield r_s^T coincides with the short-term rate. In general, the drift of N_s^T is driven by r_s^T in agreement with Theorem 8, while the one of $M_{s,t}$ exploits the stochastic rate Y_t.

7 Conclusions

This paper generalizes to stochastic-rate markets a key property of risk-neutral pricing. Indeed, if interest rates are constant (and deterministic) over time, the instantaneous rate is both the principal eigenvalue in the return-rate relation and the stochastic discount factor growth rate. If rates are stochastic, this feature is satisfied by rate-adjusted prices that, in fact, are indistinguishable from no-arbitrage prices when rates are constant. In particular, ZCB yields replace instantaneous rates and the forward measure is employed instead of the risk-neutral one. Importantly, bond yields are able to capture the growth rate of rate-adjusted pricing kernels. This rate coincides with the one of the effective pricing kernel when the horizon under consideration is infinite. Moreover, rate-adjusted pricing kernels feature a trivial transient component in their Hansen-Scheinkman decomposition. These two facts allow us to consider rate-adjusted prices as the prices of long-term interest rate risk.

The multiplicative decomposition of the no-arbitrage price of a security into a rate-adjusted price and a rate adjustment permits to disentangle the long- from the short-term exposure to interest rate risk. The rate-adjusted price is associated with persistent shocks in the term structure of rates, while the adjustment captures temporary variations in the yield curve. Therefore, our theory is fruitful for the risk management of financial contracts that feature long maturities as well as interest rate risk exposure. Beyond the applications in Section 5, our framework may shed some light on the maturity mismatch between deposits and loans in the banking system (Hoffmann et al., 2019), as well as the discounting methodology of life insurance and pension companies' assets and liabilities that are subject to the European Solvency II regulation (Jørgensen, 2018). In fact, such regulation assumes specific dynamics of rates based on the long-term yield (the Ultimate Forward Rate) dictated by European authorities every year. See the Smith and Wilson (2001) procedure.

Considering further specific dynamics of interest rates may constitute an additional avenue for future research. It could also be desirable to characterize the evolution of short rates through exogenous factors that determine the information structure, in order to assess the relation between these factors and the interest rate risk exposure over time.

Finally, from a theoretical perspective, a last challenge is to study the implications of random eigenvalues in the Perron-Frobenius theory that underlies the pricing kernel decomposition. Indeed, random dominating eigenvalues may be an indicator of non-deterministic steady states for economic dynamics.

Appendix

Appendix to *Long-term risk with stochastic interest rates*, F. Severino.

A Additional theoretical issues

Technical assumptions

In the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and is left-continuous at T. We mean that $\mathbb{F}$ is complete and right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t^+}$ for all $t \in [0, T)$, and $\mathcal{F}_T = \mathcal{F}_{T^-}$.

In the whole paper, we identify random variables that coincide almost surely and we identify stochastic processes up to modifications. Moreover, we consider processes $u : [0, T] \times \Omega \to \mathbb{R}$ that are adapted on the given filtered probability space. This requirement is equivalent to progressively measurability up to modifications (Proposition 1.12 in Karatzas and Shreve, 1991).

Forward measures

The use of different numéraires is a common practice in asset pricing: see, e.g. the comprehensive treatment by Geman et al. (1995). Regarding the T-forward measure, by Theorem 1 and Example 2 in Geman et al. (1995), the Radon-Nikodym derivative of F^T with respect to the risk-neutral measure Q on $\mathcal{F}_T$ is

$$J^T_T = \frac{dF^T}{dQ} = e^{-\int_0^T Y_\tau d\tau} \left(\mathbb{E}\left[L_T e^{-\int_0^T Y_\tau d\tau}\right]\right)^{-1} = e^{r_0^T T - \int_0^T Y_\tau d\tau}.$$

We also define, for any $t \in [0, T]$,

$$J^T_t = \mathbb{E}_t\left[L_{t,T} J^T_T\right] = e^{r_0^T T - r_t^T (T-t) - \int_0^t Y_\tau d\tau}$$

and we set $J^T_{t,T} = J^T_T / J^T_t$. The Radon-Nikodym derivative of F^T with respect to P on $\mathcal{F}_T$ is, then, $G^T_T = dF^T/dP = J^T_T L_T$ and we define, for any $t \in [0, T]$,

$$G^T_t = \mathbb{E}_t\left[G^T_T\right] = \mathbb{E}_t\left[L_T J^T_T\right] = L_t J^T_t.$$

As for t-bond yields, their relation with T-bond yields is expressed by the following compounding rule, which also ensures that $\mathbb{E}_s[M_{s,t}] = e^{-r_s^t (t-s)}$.

Lemma 9 *For any $s \leqslant t \leqslant T$, we have* $e^{r_s^T (T-s)} = e^{r_s^t (t-s)} \mathbb{E}_s^{F^T}[e^{r_t^T (T-t)}]$.

Weak time-derivative in $[s, T]$

Consider the conditional space $L^1_s(\mathcal{F}_T) = \{f \in L^0(\mathcal{F}_T) : \mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)\}$. Cerreia-Vioglio et al. (2016) show that $L^1_s(\mathcal{F}_T)$ is an L^0-module with the multiplicative decomposition $L^1_s(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$. Clearly, $L^1_s(\mathcal{F}_T)$ contains all functions f in $L^1(\mathcal{F}_T)$: in this case $\mathbb{E}_s[|f|] \in L^1(\mathcal{F}_s)$. In general, however, the conditional expectation is defined for random variables that are merely in $L^0(\mathcal{F}_T)$. See, for instance, Chapter II, §7 of Shiryaev (1996).

In $L^1_s(\mathcal{F}_T)$ we use the L^0-valued metric $d(f,g) = \mathbb{E}_s[|f-g|]$. Accordingly, we say that a stochastic process $u : [s,T] \to L^1_s(\mathcal{F}_T)$ is L^1_s-continuous if and only if, for all $t \in [s,T]$, $\mathbb{E}_s[|u_\tau - u_t|] \to 0$ a.s. when $\tau \to t$. This property is weaker than standard L^1-continuity.

Now consider the L^0-module $\mathcal{U}_s$. As a consequence of Tonelli's theorem, all processes in $\mathcal{U}_s$ are such that $\int_s^T \mathbb{E}_s[|u_\tau|]d\tau$ belongs to $L^0(\mathcal{F}_s)$, where the integral is computed trajectory by trajectory. In addition, $\mathcal{U}_s$ includes all conditional (or generalized) martingales.

We now focus on weak time-differentiability in $[s,T]$. The space $C^1_c((t,T), L^0(\mathcal{F}_s))$ employed in Definition 1 consists of functions $\varphi_s : [t,T] \to L^0(\mathcal{F}_s)$ that have compact support in (t,T) and are continuously differentiable over time in the following sense: there exists a continuous function $\psi : [t,T] \to L^0(\mathcal{F}_s)$ with compact support in (t,T) such that the pathwise integral $\int_t^\tau \psi(z)dz$ equals $\varphi_s(\tau)$ for all $\tau \in [t,T]$. For simplicity, we denote ψ by φ'_s.

The weak time-derivative in $[s,T]$ is unique, up to modifications.

Proposition 10 *Let $u \in \mathcal{U}_s$ be weakly time-differentiable in $[s,T]$. Then, the weak time-derivative of u in $[s,T]$ is unique.*

Proof. Follow the proof of Proposition 2.2 in Marinacci and Severino (2018) by replacing the unconditional expectation with the conditional expectation with respect to $\mathcal{F}_s$, and the convergence in L^1 with that in L^1_s. ■

The next result gives the relation between the weak time-derivatives in $[s_1,T]$ and $[s_2,T]$ for $s_1 \leqslant s_2$.

Proposition 11 *Let $0 \leqslant s_1 \leqslant s_2 \leqslant T$. If $u \in \mathcal{U}^1_{s_1} \cap \mathcal{U}^1_{s_2}$, then $(\mathcal{D}_{s_1}u)_t = (\mathcal{D}_{s_2}u)_t$ for every $t \in [s_2,T]$.*

Proof. Since $u \in \mathcal{U}^1_{s_1} \cap \mathcal{U}^1_{s_2}$, by Definition 1, for every $t \in [s_2,T]$,

$$\int_t^T \mathbb{E}_{s_i}\left[(\mathcal{D}_{s_i}u)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_i}(\tau)d\tau = -\int_t^T \mathbb{E}_{s_i}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'_{s_i}(\tau)d\tau$$

for all $A_t \in \mathcal{F}_t, \varphi_{s_i} \in C^1_c((t,T), L^0(\mathcal{F}_{s_i}))$ and $i = 1,2$. Since $\mathcal{F}_{s_1} \subset \mathcal{F}_{s_2}$, for any $t \in [s_2,T]$, $C^1_c((t,T), L^0(\mathcal{F}_{s_1})) \subset C^1_c((t,T), L^0(\mathcal{F}_{s_2}))$. In the following chain of equalities, we first exploit the weak time-differentiability of u in $[s_1,T]$ and then the one in $[s_2,T]$. For every $t \in [s_2,T]$, $A_t \in \mathcal{F}_t$ and $\varphi_{s_1} \in C^1_c((t,T), L^0(\mathcal{F}_{s_1}))$, we have

$$\begin{aligned}
&\int_t^T \mathbb{E}_{s_1}\left[(\mathcal{D}_{s_1}u)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_1}(\tau)d\tau = -\int_t^T \mathbb{E}_{s_1}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'_{s_1}(\tau)d\tau \\
&= -\int_t^T \mathbb{E}_{s_1}\left[\mathbb{E}_{s_2}\left[u_\tau \mathbf{1}_{A_t}\right]\right]\varphi'_{s_1}(\tau)d\tau = \mathbb{E}_{s_1}\left[-\int_t^T \mathbb{E}_{s_2}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'_{s_1}(\tau)d\tau\right] \\
&= \mathbb{E}_{s_1}\left[\int_t^T \mathbb{E}_{s_2}\left[(\mathcal{D}_{s_2}u)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_1}(\tau)d\tau\right] = \int_t^T \mathbb{E}_{s_1}\left[(\mathcal{D}_{s_2}u)_\tau \mathbf{1}_{A_t}\right]\varphi_{s_1}(\tau)d\tau.
\end{aligned}$$

The uniqueness of the weak time-derivative in $[s_1,T]$ implies that $(\mathcal{D}_{s_1}u)_t = (\mathcal{D}_{s_2}u)_t$ for every $t \in [s_2,T]$. ■

We finally prove that a weakly time-differentiable process has null weak time-derivative in $[s,T]$ if and only if it is a conditional martingale.

Proposition 12 *A process u belongs to $\mathcal{U}^1_s$ and has $\mathcal{D}_s u = 0$ if and only if it is a conditional martingale.*

Proof. Suppose that u is a conditional martingale. Then, u belongs to $\mathcal{U}_s$. Moreover, fixed $t \in [s,T]$, for any $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$ and $A_t \in \mathcal{F}_t$,

$$\int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] \varphi_s'(\tau) d\tau = \int_t^T \mathbb{E}_s\left[u_t \mathbf{1}_{A_t}\right] \varphi_s'(\tau) d\tau = \mathbb{E}_s\left[u_t \mathbf{1}_{A_t}\right] \int_t^T \varphi_s'(\tau) d\tau = 0$$

because φ_s is in $C_c^1((t,T), L^0(\mathcal{F}_s))$. Hence, $w(t) = 0$ for any $t \in [s,T]$ satisfies the definition of weak time-derivative of u in $[s,T]$ and so $\mathcal{D}_s u = 0$.

Conversely, assume that $u \in \mathcal{U}_s^1$ has $\mathcal{D}_s u = 0$. First, u is adapted and any $u_\tau \in L_s^1(\mathcal{F}_\tau)$. Therefore, $\mathbb{E}_s[|u_\tau|] \in L^0(\mathcal{F}_s)$ for all $\tau \in [s,T]$. As a consequence, $\mathbb{E}_t[|u_\tau|] \in L^0(\mathcal{F}_t)$ for all $s \leqslant t \leqslant \tau \leqslant T$. Indeed, since $|u_\tau|$ is non-negative, $\mathbb{E}_t[|u_\tau|]$ is always defined as $\mathcal{F}_t$-measurable extended random variable. However, if there existed a set $A_t \in \mathcal{F}_t$ such that $\mathbb{E}_t[|u_\tau|]\mathbf{1}_{A_t}$ equals infinity, then, taken any $B_s \in \mathcal{F}_s$ with non-empty $A_t \cap B_s$, $\mathbb{E}_s[|u_\tau|\mathbf{1}_{B_s}] \geqslant \mathbb{E}_s[\mathbb{E}_t[|u_\tau|]\mathbf{1}_{A_t \cap B_s}]$ which is infinite. This fact would contradict $u_\tau \in L_s^1(\mathcal{F}_\tau)$.

Hence, in order to prove that u is a conditional martingale, we are just left to show that u satisfies the martingale property. We begin with proving that, given $t \in [s,T]$, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take into consideration a continuous function $\eta : [t,T] \to \mathbb{R}$ with compact support in (t,T) such that $\int_t^T \eta(\tau) d\tau = 1$. Given a continuous function $\xi : [t,T] \to \mathbb{R}$ with compact support in (t,T), we define the function $k_\xi : [t,T] \to \mathbb{R}$ by $k_\xi(\sigma) = \xi(\sigma) - (\int_t^T \xi(\tau) d\tau)\eta(\sigma)$.

The function k_ξ is continuous with compact support in (t,T) and $\int_t^T k_\xi(\tau) d\tau = 0$. Thus, k_ξ has a primitive K_ξ that is continuous with compact support in (t,T). Since $K_\xi \in C_c^1((t,T), \mathbb{R})$, it is included in $C_c^1((t,T), L^0(\mathcal{F}_s))$ and so we use it as a test function in the definition of weak time-derivative of u in $[s,T]$. Since $\mathcal{D}_s u = 0$, for any $A_t \in \mathcal{F}_t$ the following holds:

$$\begin{aligned}
0 &= \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right] \left(\xi(\sigma) - \left(\int_t^T \xi(\tau) d\tau\right) \eta(\sigma)\right) d\sigma \\
&= \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right] \xi(\sigma) d\sigma - \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right] \left(\int_t^T \xi(\tau) d\tau\right) \eta(\sigma) d\sigma \\
&= \int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] \xi(\tau) d\tau - \int_t^T \left(\int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right] \eta(\sigma) d\sigma\right) \xi(\tau) d\tau \\
&= \int_t^T \left(\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right] \eta(\sigma) d\sigma\right) \xi(\tau) d\tau.
\end{aligned}$$

By Lemma A.1 in the Appendix of Marinacci and Severino (2018), for a.e. $\tau \in [t,T]$ we have $\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right] \eta(\sigma) d\sigma$. Since $\int_t^T \eta(\sigma) d\sigma = 1$,

$$\int_t^T \left(\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] - \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\right) \eta(\sigma) d\sigma = 0.$$

As the last equality is satisfied by any continuous function η with compact support in (t,T), it follows that, for a.e. $\sigma \in [t,T]$, $\mathbb{E}_s[u_\sigma \mathbf{1}_{A_t}] = \mathbb{E}_s[u_\tau \mathbf{1}_{A_t}]$ and so $\mathbb{E}_t[u_\sigma] = \mathbb{E}_t[u_\tau]$. Consequently, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$ and so $\mathbb{E}_t[u_\tau] = f_t$ for some $f_t \in L_s^1(\mathcal{F}_t)$.

u is L_s^1−right-continuous and so $\mathbb{E}_t[u_\tau]$ goes to u_t in L_s^1 when $\tau \to t^+$. Since for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[u_\tau]$ coincides a.s. with f_t, which does not depend on τ, the uniqueness of the L_s^1-limit ensures that $f_t = u_t$. Therefore, for any $t \in [0,T]$ and for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[u_\tau] = u_t$.

The last property is actually satisfied by any $\tau \in [t,T]$. Indeed, fix any τ and consider a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,T]$ such that $\tau_i \to \tau^+$ and $\mathbb{E}_t[u_{\tau_i}] = u_t$. Since u is L_s^1-right-continuous, the L_s^1-limit of $\mathbb{E}_t[u_{\tau_i}]$ is $\mathbb{E}_t[u_\tau]$. Nevertheless, $\mathbb{E}_t[u_{\tau_i}] = u_t$ for all i and so, by uniqueness of the L_s^1-limit, $\mathbb{E}_t[u_\tau] = u_t$. ■

B Proofs

Proof of Theorem 2

(Existence) In order to show that $\rho^T \in \mathcal{U}_s^1$, we prove that $e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$ and is weakly time-differentiable in $[s,T]$.

First, for all $\tau \in [s,T]$, $e^{r_s^T T}\rho_\tau^T \in L_s^1(\mathcal{F}_\tau)$. Indeed, $|e^{r_s^T T}\rho_\tau^T|$ is non-negative and so its conditional expectation at time s is an extended real random variable. However, $\mathbb{E}_s^{F^T}[|e^{r_s^T T}\rho_\tau^T|] \leqslant e^{r_s^T \tau}\mathbb{E}_s^{F^T}[|h_T|]$, which is in $L^0(\mathcal{F}_s)$. Thus, $\mathbb{E}_s^{F^T}[|e^{r_s^T T}\rho_\tau^T|]$ is in $L^0(\mathcal{F}_s)$.

Regarding L_s^1-continuity, we check that, for any $t \in [s,T)$, $\mathbb{E}_s^F[|e^{r_s^T T}\rho_\tau^T - e^{r_s^T T}\rho_t^T|]$ tends to zero when $\tau \to t^+$. We have

$$
\begin{aligned}
&\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T T}\rho_\tau^T - e^{r_s^T T}\rho_t^T\right|\right] = e^{r_s^T t}\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T(\tau-t)}\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right] \\
&\leqslant e^{r_s^T t}\Bigg(\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T(\tau-t)}\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_\tau^{F^T}[h_T]\right|\right] \\
&\quad + \mathbb{E}_s^{F^T}\left[\left|\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\Bigg) \\
&\leqslant e^{r_s^T t}\left(\left|e^{r_s^T(\tau-t)} - 1\right|\mathbb{E}_s^{F^T}[|h_T|] + \mathbb{E}_s^{F^T}\left[\left|\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\right).
\end{aligned}
$$

In the last expression, both addends go to zero a.s. when τ approaches t^+. In particular, the convergence of the first one follows from the fact that almost every realization of r_s^T is a real number (fixed for the convergence). As to the second term, its convergence is ensured by Lévy's downward theorem that guarantees that $\mathbb{E}_\tau^{F^T}[h_T]$ goes in L_s^1 to $\mathbb{E}_{t^+}^{F^T}[h_T] = \mathbb{E}_t^{F^T}[h_T]$ when $\tau \to t^+$. Similarly, when $\tau \to T^-$, the convergence is due to Lévy's upward theorem. Therefore $e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$. To be precise, as described in Chapter 14 of Williams (1991), Lévy's theorems require that $h_T \in L^1(\mathcal{F}_T)$ and ensure the previous convergences in L^1 of conditional expectations. However, these results still hold when $h_T \in L_s^1(\mathcal{F}_T)$ by using the convergence in L_s^1. Indeed, as shown in Cerreia-Vioglio et al. (2016), $L_s^1(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$ and so $h_T = a_s k_T$ for some $a_s \in L^0(\mathcal{F}_s)$ and $k_T \in L^1(\mathcal{F}_T)$. Then, Lévy's theorems apply to k_T with the convergence in L^1. The latter implies the convergence in L_s^1, which holds also after multiplying k_T by a_s and retrieving h_T.

Now we look for the weak time-derivative in $[s,T]$ of $e^{r_s^T T}\rho^T$. We consider any $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$. Since indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$,

$$
\begin{aligned}
&-\int_t^T \mathbb{E}_s^{F^T}\left[e^{r_s^T T}\rho_\tau^T \mathbf{1}_{A_t}\right]\varphi_s'(\tau)d\tau = -\int_t^T e^{r_s^T \tau}\mathbb{E}_s^{F^T}[h_T \mathbf{1}_{A_t}]\varphi'(\tau)d\tau \\
&= -\mathbb{E}_s^{F^T}[h_T \mathbf{1}_{A_t}]\int_t^T e^{r_s^T \tau}\varphi_s'(\tau)d\tau = \mathbb{E}_s^{F^T}[h_T \mathbf{1}_{A_t}]\int_t^T r_s^T e^{r_s^T \tau}\varphi_s(\tau)d\tau \\
&= \int_t^T r_s^T \mathbb{E}_s^{F^T}\left[e^{r_s^T \tau}h_T \mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau = \int_t^T \mathbb{E}_s^{F^T}\left[r_s^T e^{r_s^T T}\rho_\tau^T \mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau.
\end{aligned}
$$

The integral of the function $\sigma \mapsto e^{r_s^T \sigma}\varphi_s'(\sigma)$ is computed pathwise in $L^0(\mathcal{F}_s)$, exploiting the compact support of φ_s.

Therefore, the candidate weak time-derivative in $[s,T]$ of $e^{r_s^T T}\rho^T$ is $r_s^T e^{r_s^T T}\rho^T$. Since $r_s^T e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$, we can claim that $\mathcal{D}_s\rho^T = r_s^T\rho^T$. Of course, $\rho_T^T = h_T$ and so $\rho^T \in \mathcal{U}_s^1$ solves problem (9).

(Uniqueness) Let $f^{(1)}, f^{(2)} \in \mathcal{U}_s^1$ be two solutions of problem (9) and define $z = f^{(1)} - f^{(2)} \in \mathcal{U}_s^1$. We have that $\mathcal{D}_s z = r_s^T z$ and $z_T = 0$. We now compute the weak time-derivative of $e^{-rt}z_t$ in $[s,T]$. Fix $t \in [s,T]$. For any $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$, consider the function $\theta \mapsto e^{-r_s^T\theta} r_s^T \varphi_s(\theta) - e^{-r_s^T\theta}\varphi_s'(\theta)$ that takes values in $L^0(\mathcal{F}_s)$. By integrating pathwise, it follows that

$$\int_\tau^T \left(e^{-r_s^T\theta} r_s^T \varphi_s(\theta) - e^{-r_s^T\theta}\varphi_s'(\theta)\right) d\theta = e^{-r_s^T\tau}\varphi_s(\tau).$$

Hence, $e^{-r_s^T\tau}\varphi_s(\tau)$ belongs to $C_c^1((t,T), L^0(\mathcal{F}_s))$ and so we can use it as test function in the definition of weak time-derivative of z in $[s,T]$:

$$\begin{aligned}
&\int_t^T \mathbb{E}_s^{F^T}\left[\mathcal{D}_s z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau}\right]\varphi_s(\tau)d\tau \\
&= -\int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t}\right]\left(e^{-r_s^T\tau}\varphi_s'(\tau) - e^{-r_s^T\tau} r_s^T\varphi_s(\tau)\right)d\tau \\
&= -\int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau}\right]\varphi_s'(\tau)d\tau + \int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau} r_s^T\right]\varphi_s(\tau)d\tau.
\end{aligned}$$

Consequently, the weak time-derivative of $e^{-r_s^T t}z_t$ in $[s,T]$ is $e^{-r_s^T t}\left(\mathcal{D}_s z_t - r_s^T z_t\right)$.

However this process is null. Therefore, $e^{-r_s^T t}z_t$ has null weak time-derivative in $[s,T]$. Hence, by Proposition 12, $e^{-r_s^T t}z_t$ is a conditional F^T-martingale and so, for any $t \in [s,T]$ and $\tau \in [t,T]$, we have $\mathbb{E}_t^{F^T}[z_\tau] = e^{r_s^T(\tau - t)}z_t$. When τ goes to T^-, we get that $\mathbb{E}_t^{F^T}[z_\tau]$ tends to $e^{r_s^T(T-t)}z_t$ pointwise.

In addition, z_τ converges to $z_T = 0$ in L_s^1 as τ approaches T^- and so $\mathbb{E}_t^{F^T}[z_\tau]$ tends to zero in L_s^1. By uniqueness of the L_s^1-limit, $z_t = 0$ for all $t \in [s,T]$. This proves uniqueness of the solution of problem (9).

Proof of Proposition 3

Fix any positive s and consider the limit in probability when T goes to infinity under Assumptions 1. By Theorem 3.2 in Qin and Linetsky (2017), $r_s^T \xrightarrow{P} r^\infty$. Since r^∞ is positive, $e^{-r_s^T(T-t)} \xrightarrow{P} 0$ for all $s > 0$ and $t > s$. Therefore, $e^{-r_s^T(T-t)} - e^{-r_t^T(T-t)} \xrightarrow{P} 0$ and so $(\rho_t^T(s,h_T) - \pi_t(h_T))/\mathbb{E}_t^{F^T}[h_T]$ tends to 0.

As for the second convergence, suppose that h_T is strictly positive. Since $r_s^T \xrightarrow{P} r^\infty$ for all $s > 0$ as T goes to infinity, the difference $r_s^T - r_t^T$ converges in probability to zero for all $s > 0$ and $t > s$. In addition, for any positive ε,

$$P\left(\frac{\left|\log\rho_t^T(s,h_T) - \log\pi_t(h_T)\right|}{T-t} > \varepsilon\right) = P\left(\left|r_s^T - r_t^T\right| > \varepsilon\right)$$

and this quantity goes to zero because $r_s^T - r_t^T \xrightarrow{P} 0$ as T increases.

Proof of Proposition 4

By Lemma 9 in Appendix A, for any $t > s$

$$\mathbb{E}_s^{F^T}\left[\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)}\right] = e^{r_s^T(t-s)}\mathbb{E}_s^{F^T}\left[\frac{\pi_s(1_T)}{\pi_t(1_T)}\right] = e^{r_s^T(t-s)}\pi_s(1_t) \xrightarrow{P} e^{\left(r^\infty - r_s^t\right)(t-s)}$$

as T goes to infinity. As to the second convergence, consider the expression

$$\frac{\rho_t^T\left(s,h_T\right)}{\pi_t\left(h_T\right)}=e^{r_s^T(t-s)}\frac{\pi_s\left(1_T\right)}{\pi_t\left(1_T\right)}=e^{r_s^T(t-s)}\frac{\pi_0\left(1_{T-s}\right)}{\pi_0\left(1_{T-t}\right)}\frac{\pi_s\left(1_T\right)}{\pi_0\left(1_{T-s}\right)}\frac{\pi_0\left(1_{T-t}\right)}{\pi_t\left(1_T\right)}$$

and recall Assumptions 1. Theorem 3.2 in Qin and Linetsky (2017) ensures that, as T goes to infinity,

- $e^{r_s^T(t-s)}$ converges to $e^{r^\infty(t-s)}$ in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-s-(t-s)})$ converges to $e^{-r^\infty(t-s)}$ in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$ converges to b_s^∞ in the semimartingale topology of Émery (1979);
- $\pi_0(1_{T-t})/\pi_t(1_T)$ converges to $1/b_t^\infty$ in the semimartingale topology.

In the semimartingale topology the product of convergent processes converges to the product of the respective limit processes. Moreover, the convergence in the semimartingale topology necessarily entails the convergence in probability for any fixed t. Therefore, by Slutsky's theorem we can conclude that

$$\frac{\rho_t^T\left(s,h_T\right)}{\pi_t\left(h_T\right)}\xrightarrow{P}\frac{e^{r^\infty(t-s)}}{e^{r^\infty(t-s)}}\frac{b_s^\infty}{b_t^\infty}=\frac{b_s^\infty}{b_t^\infty},\qquad T\to+\infty.$$

Proof of Proposition 5

Fix $t\in[s,\tau]$. As we will show in Proposition 6, when T goes to infinity, $G_{t,\tau}^T$ converges in probability to $G_{t,\tau}^\infty$. Therefore, $G_{t,\tau}^T h_\tau$ goes to $G_{t,\tau}^\infty h_\tau$ in probability. Since $G_{t,\tau}^T h_\tau$ is also convergent in $L^1(P)$ and this convergence implies the one in probability, by uniqueness of the limit, $G_{t,\tau}^T h_\tau$ tends to $G_{t,\tau}^\infty h_\tau$ in $L^1(P)$. Consequently,

$$\mathbb{E}_t^{F^T}\left[h_\tau\right]=\mathbb{E}_t\left[G_{t,\tau}^T h_\tau\right]\xrightarrow{L^1}\mathbb{E}_t\left[G_{t,\tau}^\infty h_\tau\right]=\mathbb{E}_t^{F^\infty}\left[h_\tau\right],\qquad T\to+\infty$$

and the convergence is also in probability. In addition, $e^{-r_s^T(\tau-t)}\xrightarrow{P}e^{-r^\infty(\tau-t)}$ and so, by the continuous mapping theorem, when T goes to infinity,

$$\rho_t^T\left(s,h_\tau\right)=e^{-r_s^T(\tau-t)}\mathbb{E}_t^{F^T}\left[h_\tau\right]\xrightarrow{P}e^{-r^\infty(\tau-t)}\mathbb{E}_t^{F^\infty}\left[h_\tau\right]=\rho_t^\infty\left(s,h_\tau\right).$$

Proof of Proposition 6

Assumptions 1 hold and we exploit Theorem 3.2 in Qin and Linetsky (2017). Since T goes to infinity, we assume that $T>t+s$ without loss of generality.

As for the first convergence, consider

$$\frac{e^{r_s^T s}\pi_s\left(1_T\right)}{\pi_s\left(1_{T-t}\right)}=e^{r_s^T s}\frac{\pi_0\left(1_{T-s}\right)}{\pi_0\left(1_{T-t-s}\right)}\frac{\pi_s\left(1_T\right)}{\pi_0\left(1_{T-s}\right)}\frac{\pi_0\left(1_{T-t-s}\right)}{\pi_s\left(1_{T-t}\right)}.$$

When T goes to infinity, we have

- $e^{r_s^T s}$ converges to $e^{r^\infty s}$ in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-t-s})$ converges to $e^{-r^\infty t}$ in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$ converges to b_s^∞ in the semimartingale topology;
- $\pi_0(1_{T-t-s})/\pi_s(1_{T-t})$ converges to $1/b_s^\infty$ in the semimartingale topology.

As a result, the first convergence of the statement obtains. Similarly,

$$\frac{e^{-r_s^T s}\pi_s\left(1_{T-t}\right)}{\pi_t\left(1_T\right)}=e^{-r_s^T s}\frac{\pi_0\left(1_{T-t-s}\right)}{\pi_0\left(1_{T-t}\right)}\frac{\pi_s\left(1_{T-t}\right)}{\pi_0\left(1_{T-t-s}\right)}\frac{\pi_0\left(1_{T-t}\right)}{\pi_t\left(1_T\right)}$$

and, when T goes to infinity,

- $e^{-r_s^T s}$ converges to $e^{-r^\infty s}$ in probability;
- $\pi_0(1_{T-t-s})/\pi_0(1_{T-t})$ converges to $e^{r^\infty s}$ in probability;
- $\pi_s(1_{T-t})/\pi_0(1_{T-t-s})$ converges to b_s^∞ in the semimartingale topology;
- $\pi_0(1_{T-t})/\pi_t(1_T)$ converges to $1/b_t^\infty$ in the semimartingale topology.

Consequently, the second convergence of the proposition is established.

In addition, Theorem 3.2 in Qin and Linetsky (2017) ensures that $M_{s,t}=e^{-r^\infty(t-s)}(b_s^\infty/b_t^\infty)G_{s,t}^\infty$. At finite horizons we have

$$G_{s,t}^T=\left(\frac{e^{r_s^T s}\pi_s\left(1_T\right)}{\pi_s\left(1_{T-t}\right)}\right)^{-1}\left(\frac{e^{-r_s^T s}\pi_s\left(1_{T-t}\right)}{\pi_t\left(1_T\right)}\right)^{-1}M_{s,t}.$$

Since the first factor converges in probability to $e^{r^\infty(t-s)}$ and the second one to b_t^∞/b_s^∞, then $G_{s,t}^T$ converges in probability to $G_{s,t}^\infty$ when T goes to infinity.

Proof of Proposition 7

From the decomposition of $M_{s,t}$, we have that

$$N_{s,t}^T=e^{\left(r_s^T-r_t^T\right)(T-t)}e^{-r^\infty(t-s)}\frac{b_s^\infty}{b_t^\infty}G_{s,t}^\infty.$$

Here, $e^{(r_s^T-r_t^T)(T-t)}$ coincides with the ratio $\pi_t(h_T)/\rho_t^T(s,h_T)$ when an arbitrary payoff h_T is considered. Thus, by Proposition 4, it converges in probability to b_t^∞/b_s^∞ as T goes to infinity, ensuring the convergence of $N_{s,t}^T$.

Proof of Theorem 8

Problem (22). We study the properties of the process defined, at any time t, by $G_s^T e^{r_s^T s}N_{s,t}^T$ that coincides with $e^{-r_s^T t}G_t^T$. Its conditional expectation at time s under the measure P belongs to $L^0(\mathcal{F}_s)$. Moreover, L_s^1-right-continuity at t is due to the fact that

$$\mathbb{E}_s\left[\left|e^{-r_s^T\tau}G_\tau^T-e^{-r_s^T t}G_t^T\right|\right]\leqslant e^{-r_s^T t}\left(\left|e^{-r_s^T(\tau-t)}-1\right|G_s^T+\mathbb{E}_s\left[\left|\mathbb{E}_\tau\left[G_T^T\right]-\mathbb{E}_t\left[G_T^T\right]\right|\right]\right)$$

for all $\tau\geqslant t$. Similarly to the proof of Theorem 2, Lévy's downward theorem ensures the convergence to zero when $\tau\to t^+$. A parallel reasoning guarantees L_s^1-left-continuity at T and so $e^{-r_s^T t}G_t^T$ belongs to $\mathcal{U}_s$.

Next, we show that, under the physical measure, the weak time-derivative in $[s,T]$ of $e^{-r_s^T t}G_t^T$ is $-r_s^T e^{-r_s^T t}G_t^T$. By considering any $A_t\in\mathcal{F}_t$ and $\varphi_s\in C_c^1((t,T),L^0(\mathcal{F}_s))$, we have

$$\begin{aligned}&-\int_t^T\mathbb{E}_s\left[e^{-r_s^T\tau}G_\tau^T\mathbf{1}_{A_t}\right]\varphi_s'(\tau)d\tau=-\mathbb{E}_s\left[G_T^T\mathbf{1}_{A_t}\right]\int_t^T e^{-r_s^T\tau}\varphi_s'(\tau)d\tau\\&=-\mathbb{E}_s\left[G_T^T\mathbf{1}_{A_t}\right]\int_t^T r_s^T e^{-r_s^T\tau}\varphi_s(\tau)d\tau=\int_t^T\mathbb{E}_s\left[-r_s^T e^{-r_s^T\tau}G_\tau^T\mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau.\end{aligned}$$

Moving back to $N^T_{s,t}$, we showed that $N^T_{s,t}$ belongs to $\mathcal{U}^1_s$ and $\mathcal{D}N^T_{s,t} = -r^T_s N^T_{s,t}$.

Problem (23). It is convenient to consider the process defined, at any t, by $e^{r^\infty s} G^\infty_s N^\infty_{s,t}$, that is $e^{-r^\infty t} G^\infty_t$. Since G^∞_t is a martingale under P, $\mathbb{E}_s[e^{-r^\infty t} G^\infty_t] = e^{-r^\infty t} G^\infty_s$ belongs to $L^0(\mathcal{F}_s)$. In addition, $\int_s^{+\infty} \mathbb{E}_s[e^{-r^\infty \tau} G^\infty_\tau] d\tau = G^\infty_s e^{-r^\infty s}/r^\infty$ is in $L^0(\mathcal{F}_s)$, too.

L^1_s-right-continuity at any t can be shown as in the proof of problem (22) by observing that $G^\infty_\tau = \mathbb{E}_\tau[G^\infty_T]$ and $G^\infty_t = \mathbb{E}_t[G^\infty_T]$ for any T larger than τ. As a result, $e^{-r^\infty t} G^\infty_t$ is in $\mathcal{U}_s$ with $T = +\infty$.

Regarding weak time-differentiability in $[s, +\infty)$, we can follow again the proof of problem (22) by integrating on intervals $[t, +\infty)$. Indeed, it is enough to use the relation $G^\infty_\tau = \mathbb{E}_\tau[G^\infty_T]$, where T is a time index larger than any instant in the (bounded) supports of φ_s and φ'_s. Thus, the weak time-derivative in $[s, +\infty)$ of $e^{-r^\infty t} G^\infty_t$ is $-r^\infty e^{-r^\infty t} G^\infty_t$. Consequently, $N^\infty_{s,t}$ turns out to be a process in $\mathcal{U}^1_s$ with $T = +\infty$ that satisfies $\mathcal{D}N^\infty_{s,t} = -r^\infty N^\infty_{s,t}$.

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