

# Long-term risk with stochastic interest rates\*

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## Abstract

Investors with heterogeneous trading horizons require compensation for the exposure to different risks. The no-arbitrage valuation over increasing horizons is described by the evolution of stochastic discount factors (SDFs). Each of them exhibits a multiplicative decomposition into deterministic growth term, permanent and transient component, provided by Hansen and Scheinkman (2009). In particular, the growth rate captures the deterministic discounting for risks that are relevant in the long term. When interest rates in the market are constant, the SDF growth rate coincides with the instantaneous rate. On the contrary, when rates of interest are stochastic, the SDF growth rate is given by the long-term yield of zero-coupon bonds, which is unsuitable for instantaneous no-arbitrage valuation.

We show how to reconcile the long-run properties of the SDF with the instantaneous relations between returns and rates in stochastic-rate markets. In particular, we introduce a rate adjustment in pricing that isolates the short-term variability of rates. No-arbitrage prices are then factorized into rate-adjusted prices and a rate adjustment that is absent when interest rates are constant. Rate-adjusted prices employ constant yields to maturity for discounting future payoffs over time. The rate-adjusted SDF features the same long-term growth rate of the SDF in the market but has no transient component in its Hansen-Scheinkman decomposition. Therefore, rate-adjusted prices provide the proper valuation for long-term interest rate risk. Moreover, we show how this novel notion is fruitful for managing the interest rate risk related to fixed-income derivatives, life insurances and annuities.

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## 1 Motivations and main results

A deep analysis of the term structure of interest rates is essential for dynamic asset pricing theory. Once fixed-income securities are issued by an institution, they are likely to display time-varying yields to maturity in the secondary market. In fact, different yields embody the changeable expectations of market participants about future events that could impinge on the credit market at different horizons. Moreover, the relation between short-term and long-term bond yields is an indicator of the overall performance of the economy.<sup>1</sup> In general, understanding the consistent aggregation of interest rate risks associated with heterogeneous maturities has been a challenge of the last century.<sup>2</sup>

Similarly, in dynamic asset pricing theory, investors with different trading horizons require compensation for the exposures to sources of randomness with heterogeneous duration. As a consequence, long-term premia are the outcome of the intertemporal aggregation of local risk exposures. Importantly, the components of economic variables that survive in the long run are free from inaccuracies due to temporary risk adjustments and are likely to reflect economic fundamentals. For example, Hansen, Heaton, and Li (2008) provide a theoretical analysis of risk-return trade-offs across increasing maturities.

Hansen and Scheinkman (2009) develop a general operator framework to connect short- and long-run risk compensations. Their valuation of cashflows exploits the evolution of both stochastic growth and stochastic discount functionals. In particular, in arbitrage-free markets every stochastic discount factor features a multiplicative decomposition in a deterministic growth term, a permanent (or martingale) component and a transient factor. The martingale component induces a probability measure change in the long run and is responsible for a large variance explanation. In equilibrium this decomposition impacts the asymptotic dynamics of aggregate consumption and wealth through investors' marginal utility, as assessed by Alvarez and Jermann (2005) and Hansen (2012).

Stochastic discount factors reflect the risk-neutral valuation of state-contingent payoffs in viable markets. In particular, the deterministic growth rate elicited by the Hansen-Scheinkman decom-

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<sup>1</sup>For instance, inverted yield curves may constitute a predictor for recessions (see Ang, Piazzesi, and Wei, 2006).

<sup>2</sup>A substantial stream of economic literature involves the expectation theory, introduced by Fisher (1930) and developed by Keynes (1930). Subsequent elaborations are due to Lutz (1940) and Hicks (1946), among the others. See the surveys by Cox, Ingersoll, and Ross (1985) and Russell (1992).

position provides a proper discounting adjusted for risks that are relevant in the long run. When interest rates in the market are constant over time, the pricing kernel growth rate coincides with the same instantaneous rate of interest at any maturity under scrutiny. When rates are stochastic, their variability is captured by the interplay of stochastic discounting and stochastic growth, as discussed in Borovička, Hansen, Hendricks, and Scheinkman (2011). Unfortunately, in this case it is hard to achieve a synthetic characterization of pricing kernel growth rates across increasing horizons.

This work fills this gap and builds a bridge between fixed-income and long-term risk literatures, showing the extent to which the intertemporal aggregation of instantaneous random rates contributes to pricing kernel growth rates at increasingly large horizons. Moreover, by shifting the maturity farther and farther over time, we show how to retrieve the asymptotic deterministic growth rate inferred by Qin and Linetsky (2017). Notably, our results do not depend on the specific dynamics of interest rates.

## 1.1 Main results

We consider a continuous-time arbitrage-free market in the time interval  $[0, T]$ , composed of both risky and fixed-income securities that depend on a stochastic instantaneous rate  $Y_t$ . We concentrate on price dynamics, while simultaneously focusing on the evolution of stochastic discount factors. We develop the theory in a conditional setting.

Regarding price dynamics, we start considering the no-arbitrage price  $\pi_t(h_T)$  at time  $0 \leq t \leq T$  of an attainable payoff with maturity  $T$ . When rates of interest are constant over time, the no-arbitrage condition leads to the equality

$$\text{instantaneous asset return} = \text{instantaneous risk-free rate}$$

under a risk-neutral measure  $Q$ . In the language of Hansen and Scheinkman (2009) and Hansen (2012), this relation is formalized by the eigenvalue-eigenvector problem

$$\mathcal{A}\pi_t = r \pi_t, \quad t \in [0, T]. \tag{1}$$

Here,  $\mathcal{A}$  is the extended infinitesimal generator of the Markov price process.  $\mathcal{A}$  works as a differential operator and generalizes the ordinary differential equation satisfied by risk-free bonds with interest rate  $r$ . Importantly, the eigenvalue  $r$  has a prominent role in the evolution of the stochastic discount factor since it captures its deterministic growth rate. Indeed, the pricing kernel in any time interval

$[s, t]$  is  $M_{s,t} = e^{-r(t-s)} L_{s,t}$ , where  $L_{s,t}$  is the conditional Radon-Nikodym density of  $Q$  with respect to the physical probability  $P$ .

When rates are constant over time, the instantaneous rate determines both the risk-neutral price dynamics and the pricing kernel growth rate, irrespective of the time interval under consideration. In a stochastic-rate setting, the action of instantaneous rates is more convoluted. Indeed, while eq. (1) may be rephrased at any instant  $t$  by using the random rate  $Y_t$  instead of  $r$ , the sole instantaneous rate  $Y_t$  is unable to subsume the stochastic discount factor growth rate on any given time period. In particular, the pricing kernel in  $[s, t]$  takes the form  $M_{s,t} = e^{-\int_s^t Y_\tau d\tau} L_{s,t}$ .

Comparing the expressions of  $M_{s,t}$  in the two contexts, we obtain an indication of the heavy measurability requirement of the growth term  $e^{-\int_s^t Y_\tau d\tau}$  with respect to the deterministic  $e^{-r(t-s)}$ . The definition of a growth rate for  $M_{s,t}$  is indeed challenging when rates of interest are floating over time. Under regularity conditions, Qin and Linetsky (2017) individuate a deterministic *long-term yield* as the asymptotic growth rate of the stochastic discount factor. Nevertheless, the quantity identified by Qin and Linetsky – which is the limit of bonds yields at infinite horizons – is not suitable for characterizing instantaneous returns in the sense of eq. (1). The conceptual reason beyond these difficulties rests upon the dual nature of stochastic interest rates. On the one hand, they represent a proxy for the (absent) risk-free rates. On the other, they constitute a source of randomness per se.

The main contribution of this paper is to provide a generalization of eq. (1) for stochastic-rate settings where the employed eigenvalue is determinant for the pricing kernel growth rate. The previously described relations for rates that are constant over time follow as a special case of our construction. In addition, our generalization is consistent with the asymptotic results of the long-term risk literature.

Our theory is based on the introduction of *rate-adjusted prices* that provide hedging from interest rate variability. In general, given an attainable payoff  $h_T$  at time  $T$ , we denote by  $\pi_t(h_T)$  its no-arbitrage price at  $t$ . For example,  $\pi_t(1_T)$  is the no-arbitrage price of a pure discount  $T$ -bond. The rate-adjusted price of  $h_T$  at time  $t$  in the interval  $[s, T]$ , denoted by  $\rho_t^T(s, h_T)$ , satisfies

$$\underbrace{\rho_t^T(s, h_T)}_{\text{rate-adjusted price}} = \underbrace{e^{r_s^T(t-s)} \frac{\pi_s(1_T)}{\pi_t(1_T)}}_{\text{adjustment}} \cdot \underbrace{\pi_t(h_T)}_{\text{no-arbitrage price}}, \quad (2)$$

where  $r_s^T$  is the yield-to-maturity of a zero-coupon  $T$ -bond at time  $s$ . If rates of interest are constant, rate-adjusted and no-arbitrage prices coincide. In general, the two prices are equal at the instants  $s$  and  $T$ .

Interestingly,  $\rho^T$  can be interpreted as an indifference price for an investor that ignores the variability of rates, or as the conversion price of a convertible bond (when  $h_T$  refers to a stock). In particular, the rate-adjusted price of a zero-coupon bond is the no-arbitrage price of the same security in a parallel market with constant interest rates equal to the yield  $r_s^T$ . Moreover, when the horizon  $T$  goes to infinity, the ratio between  $\rho_t^T$  and  $\pi_t$  is convergent both in probability and in expectation.

To look at the matter from the standpoint of stochastic discount factors, we define the *rate-adjusted pricing kernel* in the interval  $[s, t]$  with  $s \leq t \leq T$  by

$$N_{s,t}^T = e^{-r_s^T(t-s)} \frac{\pi_t(1_T)}{\pi_s(1_T)} M_{s,t}.$$

As expected,  $N_{s,t}^T$  and  $M_{s,t}$  coincide when interest rates are constant.

To formalize our differential pricing relations in a conditional setting, we introduce a mathematical instrument that allows us to differentiate stochastic processes on the time window  $[s, T]$  by disentangling the role of known information at instant  $s$ : the weak time-derivative in  $[s, T]$ , denoted by  $\mathcal{D}$ . This tool is a generalization of the weak time-derivative of Marinacci and Severino (2018) that applies to a wide class of semimartingale processes and extends the infinitesimal generator. Similarly to the original notion, the weak time-derivative in  $[s, T]$  provides a handy characterization of conditional martingales and so it is useful for dealing with discounted no-arbitrage prices and forward prices. These processes feature, indeed, a null weak time-derivative in  $[s, T]$ , which conveniently generalizes the zero-drift condition of continuous-time asset pricing to a wide class of semimartingales.

Our main results are Theorem 2 in Section 3 and Theorem 9 in Section 4. We prove that rate-adjusted prices and rate-adjusted pricing kernels satisfy the differential relations

$$\mathcal{D}\rho_t^T = r_s^T \rho_t^T, \quad \mathcal{D}N_{s,t}^T = -r_s^T N_{s,t}^T \tag{3}$$

for any  $t$  in  $[s, T]$ , where the first equality holds under the forward measure and the second one under the physical measure. These equations parallel the relations satisfied by no-arbitrage prices and by the pricing kernel  $M_{s,t}$  when interest rates are constant:

$$\mathcal{D}\pi_t = r\pi_t, \quad \mathcal{D}M_{s,t} = -rM_{s,t},$$

where the two equalities hold under the risk-neutral measure  $Q$  and the measure  $P$ , respectively. The equations on the left rephrase the relation between returns and rates, while the parameters in

the right-hand side equalities identify the pricing kernel growth rates.

The constant rate is replaced by the yield  $r_s^T$ , conveying the intuition that pure discount bonds play the role of risk-free assets in a context in which the money market account is unforeseeable. The results are robust when the horizon is moved to infinity, as we discuss in Subsection 3.6. In this case, the long-term yield arises as the growth rate for both the rate-adjusted pricing kernel and  $M_{s,t}$ . Indeed, the long-term rate-adjusted pricing kernel  $N_{s,t}^\infty$  differs from  $M_{s,t}$  only in the transient component in the Hansen and Scheinkman (2009) decomposition. Such component is trivial only for  $N_{s,t}^\infty$  when interest rates are stochastic. Hence, pricing with the rate adjustment means overlooking the transient component in the pricing kernel and, therefore, focusing on the long-term interest rate risk.

The dynamics of eq. (3) are conditional on the information structure available at date  $s$ . We are actually considering an eigenvalue-eigenvector problem in which the eigenvalue is a random variable, known at the beginning of the trading interval. This approach combines the intuition of conditional asset pricing theory – that dates back to Hansen and Richard (1987) – with the Perron-Frobenius theory applied to economic theory by Hansen and Scheinkman (2009).

As we discuss in the body of the paper, results similar to eq. (3) are unlikely to hold for no-arbitrage prices when rates are stochastic: the instantaneous rate  $Y_t$  cannot serve as eigenvalue and the transitory component of  $M_{s,t}$  hinders the differentiation. Nonetheless, rate-adjusted prices are able to reach the purpose. Indeed, pure discount bond yields play both the role of eigenvalues (for rate-adjusted prices) and that of growth rates (for rate-adjusted pricing kernels), generalizing the features of the constant-rate case. Table 1 at the end of Section 4 summarizes our results.

Eq. (2) is actually a decomposition of the no-arbitrage price into a rate-adjusted price (sensible to long-lasting shocks in the yield curve) and an adjustment (reacting to temporary variations of rates). We are providing a way to disentangle the short- and long-term interest rate risk associated with a given cashflow. Therefore, our theory has promising applications to the risk management of financial products and insurance contracts with distant maturities and significant exposure to interest rate variability. In Section 5 we provide a bundle of applications of rate-adjusted valuation to derivative pricing (interest rate swaps), life insurance actuarial present values, mortgage renegotiations and pension fund withdrawals. The different discounting ways related to no-arbitrage and rate-adjusted pricing permit to quantify the exposure to short- and long-term interest rate variations.

Section 6 illustrates the theory in the context of single-factor affine interest rate models. The Appendix to the paper discusses several theoretical issues and provides all the proofs, additional simulations and an application of rate-adjusted valuation to optimal consumption problems.

## 2 Asset pricing framework

We start with describing a general continuous-time arbitrage-free market with stochastic interest rates. We then focus on the conditions that ensure long-term convergences, we describe the forward measures and we discuss the main properties of weak time-derivatives. All technical details are collected in Appendix A.

### 2.1 Arbitrage-free market

We fix  $T > 0$  and consider a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , where the filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  satisfies the usual conditions and is left-continuous in  $T$ . We build a financial market by considering two adapted random processes  $X = \{X_t\}_{t \in [0, T]}$  and  $Y = \{Y_t\}_{t \in [0, T]}$ . The  $N$ -dimensional process  $X$  consists of the prices of primary risky assets that generate the market. The one-dimensional process  $Y$  represents the instantaneous interest rate and the process  $\{e^{\int_0^t Y_\tau d\tau}\}_{t \in [0, T]}$  denotes the money market account. Pure discount bonds with any possible maturity are traded, too. For simplicity the face value of these instruments is one unit.

At any instant  $t$  we define the vector of relative prices  $Z_t = e^{-\int_0^t Y_\tau d\tau} X_t$  obtained by discounting asset prices by the value of the money market account. We assume that our price system satisfies the *no free lunch with vanishing risk* condition and that relative asset prices are semimartingales. Therefore, by the First Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer (1998) there exists a probability measure equivalent to  $P$  such that  $Z$  is a sigma-martingale.<sup>3</sup> We assume that at least one of these sigma-martingale measures, denoted by  $Q$ , is actually an equivalent martingale measure. We indicate by  $L_T$  the Radon-Nikodym derivative of  $Q$  with respect to  $P$  and we set  $L_t = \mathbb{E}_t[L_T]$  for all  $t \in [0, T]$ , where  $\mathbb{E}_t$  is the conditional expectation with respect to  $\mathcal{F}_t$  under  $P$ . We also define  $L_{t,T} = L_T/L_t$ .

We denote by  $M = \{M_t\}_{t \in [0, T]}$  the strictly positive stochastic discount factor process associated with the measure  $Q$ , namely  $M_t = e^{-\int_0^t Y_\tau d\tau} L_t$  for all  $t$ . In addition, we define the pricing kernel in any time interval  $[s, t]$  with  $s \leq t \leq T$  by  $M_{s,t} = M_t/M_s = e^{-\int_s^t Y_\tau d\tau} L_{s,t}$ .

The no-arbitrage price at time  $t$  of a zero-coupon bond with redemption date  $T$  is  $\pi_t(1_T)$  and the related *yield to maturity* is  $r_t^T = -\log \pi_t(1_T)/(T - t)$ .<sup>4</sup> The yield  $r_t^T$  may be interpreted as an average rate of interest which is ex-ante equivalent to the compounding of all forthcoming instantaneous rates  $Y_\tau$  when  $\tau$  spans the interval  $[t, T]$ . Clearly, if interest rates are constant over

<sup>3</sup>A sigma-martingale is the martingale transform of a martingale by an integrable predictable process (Émery, 1980).

<sup>4</sup>We also define  $r_T^T$  as the a.s. limit of  $r_t^T$  when  $t$  approaches  $T$ .

time (and deterministic), the yield coincides with the instantaneous rate.

## 2.2 Long-term assumptions

Our semimartingale framework is compatible with the setting of Qin and Linetsky (2017) that formalize the convergence of bond yields, forward measures and stochastic discount factors when the horizon  $T$  becomes larger and larger.

**Assumptions 1** *We assume the following.*

- $M_t$  is a strictly positive semimartingale such that  $\mathbb{E}[M_{t,T}]$  is finite for all  $0 \leq t < T$ .
- For all  $t > 0$ , when  $T$  goes to infinity,  $\mathbb{E}_t[M_T]/\mathbb{E}[M_T]$  converges in  $L^1$  to a positive  $\mathcal{F}_t$ -measurable random variable  $G_t^\infty$ .
- For all  $t > 0$ , when  $T$  goes to infinity,  $\pi_0(1_{T-t})/\pi_0(1_T)$  has a positive finite limit.
- For all  $t > 0$ , when  $T$  goes to infinity, the limit in probability of  $r_t^T$  is positive.

The three first assumptions are from Qin and Linetsky (2017) and ensure that, for any  $t$ , the yield to maturity  $r_t^T$  converges in probability to a deterministic *long-term yield*  $r^\infty$  (see Theorem 3.2 therein). This result is consistent with the persistence of the yield curve over large maturities widely documented empirically and discussed, e.g., in Diebold and Li (2006). Notably,  $r^\infty$  is not dependent on  $t$ , consistently with the impossibility of falling long-term rates illustrated by Dybvig, Ingersoll Jr, and Ross (1996). The fourth assumption additionally ensures that  $r^\infty$  is positive.

The long-term yield is associated with a long bond, obtained by a roll-over portfolio of zero-coupon bonds across increasing maturities. The time  $t$  value of this portfolio is denoted by  $B_t^\infty$ , while  $b_t^\infty$  is the related long-term discounted value, namely  $b_t^\infty = e^{-r^\infty t} B_t^\infty$ .

## 2.3 Forward measures and pricing

By using as numéraire the no-arbitrage price of a zero-coupon  $T$ -bond, we construct the forward measure with horizon  $T$  or, simply,  *$T$ -forward measure* and we denote it by  $F^T$ . See e.g. Geman, El Karoui, and Rochet (1995). This probability measure is equivalent to  $Q$  and we indicate its Radon-Nikodym derivative with respect to  $P$  by  $G_T^T$ . Moreover, we set  $G_t^T = \mathbb{E}_t[G_T^T]$  for any  $t \in [0, T]$  and we define  $G_{t,T}^T = G_T^T/G_t^T$ . In the special case in which rates of interest are constant over time,  $F^T$  coincides with  $Q$ .



Using  $G_t^T$ , the stochastic discount factor and the pricing kernel in any interval  $[s, t]$  may be expressed as

$$M_t = e^{r_t^T(T-t) - r_0^T T} G_t^T, \quad M_{s,t} = e^{r_t^T(T-t) - r_s^T(T-s)} G_{s,t}^T.$$

Although  $M_{s,t}$  refers to the time window  $[s, t]$ , its expression depends on the ultimate horizon  $T$  through the density of the  $T$ -forward measure.

In our setting the  $T$ -forward measure provides a handy representation of risk-neutral asset prices. Consider, for instance, an attainable  $\mathcal{F}_T$ -measurable payoff  $h_T$  such that  $\mathbb{E}_s^{F^T} [|h_T|]$  is  $\mathcal{F}_s$ -measurable. Its no-arbitrage price at any time  $t \in [s, T]$  can be written in equivalent ways, depending on the numéraire change:

$$\pi_t(h_T) = \mathbb{E}_t^Q \left[ e^{-\int_t^T Y_\tau d\tau} h_T \right] = e^{-r_t^T(T-t)} \mathbb{E}_t^{F^T} [h_T]. \quad (4)$$

The right-hand side of this expression actually makes a fruitful bridge between constant-rate and stochastic-rate asset valuation.

We now consider the case in which the horizon  $T$  becomes arbitrarily large under the Assumptions 1. Qin and Linetsky (2017) prove in Theorem 3.1 that  $F^T$  strongly converges to the *long-term forward measure*  $F^\infty$  when  $T$  goes to infinity. At any time  $t$  the Radon-Nikodym derivative of  $F^\infty$  with respect to  $P$  is  $G_t^\infty$  and the collection of all  $G_t^\infty$  constitutes a  $P$ -martingale process. In addition, we set  $G_{s,t}^\infty = G_t^\infty / G_s^\infty$  for all  $s$  and  $t$ . Interestingly, the long-term forward measure is related to a specific numéraire that is exactly the price  $B_t^\infty$  of the long bond.

Finally, Qin and Linetsky (2017) prove in Theorem 3.2 that the pricing kernel  $M_{s,t}$  satisfies the long-run decomposition

$$M_{s,t} = e^{-r^\infty(t-s)} \frac{b_s^\infty}{b_t^\infty} G_{s,t}^\infty. \quad (5)$$

In a Markovian environment,  $G_{s,t}^\infty$  defines the *permanent* (or *martingale*) *component* of Hansen and Scheinkman (2009) decomposition, while  $b_s^\infty / b_t^\infty$  constitutes the *transient component* and  $r^\infty$  is the deterministic *long-term growth rate*.<sup>5</sup> Furthermore, from eq. (5) it is easy to write the no-arbitrage price of any payoff  $h_T$  by using the long-term forward measure:

$$\pi_t(h_T) = e^{-r^\infty(T-t)} \mathbb{E}_t^{F^\infty} \left[ \frac{b_t^\infty}{b_T^\infty} h_T \right]. \quad (6)$$

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<sup>5</sup>For the empirical estimation of these terms see Christensen (2017) and Qin, Linetsky, and Nie (2018). A non-Markovian example is provided by Qin and Linetsky (2018).

## 2.4 Weak time-derivative in $[s, T]$

In order to analyze the increments of stochastic processes we need a differential operator. Marinacci and Severino (2018) introduce the *weak time-derivative* that applies to a wide class of semimartingales and generalizes the infinitesimal generator for Markov processes. Here we extend this notion to a conditional setting. Appendix A collects the most technical details and the proofs.

We fix an instant  $s \in [0, T]$  and we consider the conditional space  $L_s^1(\mathcal{F}_T)$  composed of variables  $f \in L^0(\mathcal{F}_T)$  such that  $\mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)$  and endowed with the  $L^0$ -valued metric  $d(f, g) = \mathbb{E}_s[|f - g|]$ . Moreover, we denote by  $\mathcal{U}_s$  the  $L^0$ -module of adapted processes  $u : [s, T] \rightarrow L_s^1(\mathcal{F}_T)$  that are  $L_s^1$ -right-continuous in  $[s, T)$  and  $L_s^1$ -left-continuous in  $T$ . We then define the weak time-derivative for processes in  $\mathcal{U}_s$ . For any  $t \in [s, T]$  the definition uses the space  $C_c^1((t, T), L^0(\mathcal{F}_s))$  of functions  $\varphi_s : [t, T] \rightarrow L^0(\mathcal{F}_s)$  that have compact support in  $(t, T)$  and are continuously differentiable over time.<sup>6</sup>

**Definition 1** *We say that a process  $u \in \mathcal{U}_s$  is weakly time-differentiable in  $[s, T]$  when there exists a process  $v \in \mathcal{U}_s$  such that for every  $t \in [s, T]$*

$$\int_t^T \mathbb{E}_s [v_\tau \mathbf{1}_{A_t}] \varphi_s(\tau) d\tau = - \int_t^T \mathbb{E}_s [u_\tau \mathbf{1}_{A_t}] \varphi_s'(\tau) d\tau$$

for all  $A_t \in \mathcal{F}_t$  and  $\varphi_s \in C_c^1((t, T), L^0(\mathcal{F}_s))$ . In this case, we call  $v$  a weak time-derivative of  $u$  in  $[s, T]$ .

Definition 1 generalizes the weak time-derivative of Definition 2.1 in Marinacci and Severino (2018), where  $s = 0$  and deterministic test functions are employed. The weak time-derivative in  $[s, T]$  is unique (Proposition 11 in Appendix A) and we denote it by  $\mathcal{D}u$ . Then, we introduce the  $L^0$ -submodules of  $\mathcal{U}_s$ , denoted by  $\mathcal{U}_s^1$  and  $\mathcal{U}_s^\infty$ , that consist of weakly time-differentiable (or infinitely weakly time-differentiable) processes in  $[s, T]$ .

$\mathcal{F}_s$ -measurable functions play the role of multiplicative constants for the weak time-derivative in  $[s, T]$ . Indeed, given a process  $u \in \mathcal{U}_s^1$  and  $\xi_s \in L^0(\mathcal{F}_s)$ , the process defined by  $\xi_s u_t$  for all  $t \in [s, T]$  belongs to  $\mathcal{U}_s^1$ , too, and  $\mathcal{D}(\xi_s u) = \xi_s \mathcal{D}u$ . This property allows us to deal with  $\mathcal{F}_s$ -measurable parameters in the differential equations and to study eigenvalue-eigenvector problems for  $\mathcal{D}$  with  $\mathcal{F}_s$ -measurable eigenvalues.

The key feature of weak time-derivatives in  $[s, T]$  is the characterization of *conditional* (or

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<sup>6</sup>The integrals in the next definition are pathwise integrals of processes in  $L^0(\mathcal{F}_s)$ .

generalized) martingales.<sup>7</sup> Importantly, a process belongs to  $\mathcal{U}_s^1$  and has null weak time-derivative in  $[s, T]$  if and only if it is a conditional martingale (Proposition 12 in Appendix A).

When the risk-neutral probability  $Q$  is considered, a simple example of conditional martingale is provided by the process of *future prices* of a claim with expiry  $T$  at any  $t$  in  $[s, T]$ . On the contrary, under the forward measure, *forward prices* in  $[s, T]$  for the settlement date  $T$  are conditional martingales. See, for instance, Sections 9.6 and 11.5 in Musiela and Rutkowski (2005). Both these processes exhibit null weak time-derivatives in  $[s, T]$  with respect to different measures.<sup>8</sup>

### 3 Pricing equation

We formulate and solve a *rate-adjusted* pricing equation for the valuation of random payments in a market with stochastic interest rates. We, then, compare the solution of the equation with the usual risk-neutral pricing formula. We interpret rate-adjusted prices as indifference prices (that hedge from interest rate randomness) and as conversion prices for hybrid securities. We finally study their properties in the long run and we describe the valuation of cashflows. Proofs can be found in Appendix B.

#### 3.1 Rate-adjusted and no-arbitrage pricing

In the time interval  $[s, T]$  we face the problem of evaluating an  $\mathcal{F}_T$ -measurable payoff  $h_T$ . As illustrated in Subsection 2.3, we can express the time  $t$  no-arbitrage price of  $h_T$  by using both the measure  $Q$  and the forward measure. Up to replacing the risk-neutral measure with  $F^T$  and instantaneous rates with bond yields, the right-hand side of eq. (4) formally traces no-arbitrage pricing with constant rates of interest.

As discussed by Hansen and Scheinkman (2009), in an arbitrage-free market with constant rate of interest  $r$ , asset prices satisfy the relation

$$\mathcal{A}\pi = r \pi, \tag{7}$$

where  $\mathcal{A}$  is the extended generator of an underlying Markov process and the risk-neutral measure is employed. This eigenvalue-eigenvector problem has its roots in the Perron-Frobenius theory and

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<sup>7</sup>By this terminology we mean processes  $u$  defined in the time interval  $[s, T]$  with all the properties of martingales except for integrability, which is replaced by the weaker condition  $\mathbb{E}_t[u_\tau] \in L^0(\mathcal{F}_t)$  for all  $s \leq t \leq \tau \leq T$ . See, for instance, Chapter VII, §1 of Shiryaev (1996).

<sup>8</sup>The weak time-derivative in  $[s, T]$  is also useful for the analysis of no-arbitrage prices because, after discounting by the money-market account, they are martingales under  $Q$ . Moreover, the weak time-derivative captures the drift of semimartingale processes and applies to a wide range of continuous-time asset pricing models. See Marinacci and Severino (2018) and the example in Subsection 3.1.

captures the essence of no-arbitrage. Indeed, it relates infinitesimal price increments of a possibly risky security with the interest rate deriving from a locally riskless investment. This approach is known in the financial literature since Cox and Ross (1976) derivation of Black-Scholes formula via a hedging portfolio. Moreover, the rate  $r$ , up to a sign change, defines the growth rate of the stochastic discount factor in the celebrated Hansen-Scheinkman decomposition, independently on the time horizon under consideration.

Marinacci and Severino (2018) formalize the eigenvalue-eigenvector problem of eq. (7) in a semimartingale framework by using weak time-derivatives on the time interval  $[0, T]$  with the measure  $Q$ . In particular, they show that the risk-neutral price process given by  $\pi_t(h_T)$  is the unique solution in  $\mathcal{U}_0^1$  of the no-arbitrage pricing equation

$$\begin{cases} \mathcal{D}f_t = r f_t & t \in [0, T] \\ f_T = h_T \end{cases} \quad (8)$$

where  $h_T \in L^1(\mathcal{F}_T, Q)$ . In this problem the interest rate  $r$  is supposed to be constant.

**Example.** As an illustrative example we consider a Black and Scholes (1973) market with a riskless bond and a risky security whose prices follow the dynamics

$$dB_t = rB_t dt, \quad dX_t = \mu X_t dt + \sigma X_t dW_t^P$$

under the physical measure. Here,  $r, \mu \in \mathbb{R}$ ,  $\sigma > 0$  and  $W_t^P$  is a standard Wiener process. It is apparent that the bond price satisfies problem (8). As for the risky asset, under the physical measure the Wiener process  $W_t^P$  exhibits null weak time-derivative because it is a martingale, but the drift of  $X_t$  differs from  $rX_t$ . However, moving to the risk-neutral measure  $Q$ , the dynamics of  $X_t$  becomes

$$dX_t = rX_t dt + \sigma X_t dW_t^Q, \quad (9)$$

where  $W_t^Q$  is a  $Q$ -Wiener process, and so  $\mathcal{D}X = rX$ . Hence, problem (8) is satisfied. In this arbitrage-free market both the risk-free and the risky security share the same drift, determined by the instantaneous rate.

The dynamics presented so far generalize to stochastic-rate settings by replacing  $r$  with the instantaneous rate  $Y_t$ . For example, Chapter 1 of Karatzas and Shreve (1998) shows how to obtain similar dynamics to eq. (9) in a semimartingale pricing framework with random rates. The same approach is exploited by Heath, Jarrow, and Morton (1992) in deriving no-arbitrage restrictions on the forward rate drift. Nevertheless, there are two major flaws in these generalizations. First,

the drift coefficient  $Y_t$  is unable to capture the stochastic discount factor growth rate over any time period, differently from the constant-rate case. Second,  $Y_t$  is a random process. Hence, it is not known ex ante (since it is floating over time) and so it forbids an eigenvalue-eigenvector formulation of the problem in the spirit of Hansen and Scheinkman (2009).

Our generalization of problem (8) solves these issues for any given dynamics of interest rates. We employ the forward measure instead of  $Q$  and  $T$ -bond yields in place of short-term rates to formulate a suitable eigenvalue-eigenvector problem. Moreover, we provide a conditional version of the differential problem defined on any time window  $[s, T]$  by using the weak time-derivative in  $[s, T]$ . The arising eigenvalue is an  $\mathcal{F}_s$ -measurable random variable (known at the beginning of the trading interval) and, consistently, the growth rate of the stochastic discount features the same  $\mathcal{F}_s$ -measurability.

To enter the details of our construction, we place ourselves in the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, F^T)$ , where we employ the forward measure. We solve in  $\mathcal{U}_s^1$  the following pricing differential equation with random coefficient given by the yield to maturity  $r_s^T \in L^0(\mathcal{F}_s)$ , i.e.

$$\begin{cases} \mathcal{D}f_t = r_s^T f_t & t \in [s, T) \\ f_T = h_T \end{cases} \quad (10)$$

with  $h_T \in L_s^1(\mathcal{F}_T, F^T)$ . We refer to (10) as the *rate-adjusted pricing equation*.

**Theorem 2** *There exists a unique solution of problem (10) in  $\mathcal{U}_s^1$ , given by*

$$\rho_t^T(s, h_T) = e^{-r_s^T(T-t)} \mathbb{E}_t^{F^T} [h_T] \quad \forall t \in [s, T]. \quad (11)$$

We refer to  $\rho_t^T(s, h_T)$  as the *rate-adjusted price* of  $h_T$  at time  $t$  in the interval  $[s, T]$ .

Theorem 2 is substantially an assessment of the conditional martingale property of the process  $\{e^{r_s^T(T-t)} \rho_t^T\}_{t \in [s, T]}$  under  $F^T$ . This process is, in fact, a collection of forward prices for  $h_T$ . Under specific assumptions on asset and rate dynamics, the problem can be tackled from the perspective of the Feynman-Kac partial differential equation, where null weak time-derivatives boil down to the zero-drift condition (see Subsection 6.2.1).

At any instant  $t$  in  $[s, T]$ ,  $r_s^T$  is the only average rate employed by  $\rho_t^T$  for the valuation on  $h_T$ . The valuation instant is synchronous with the information set only for the initial  $\rho_s^T$ . In particular,  $\rho_s^T$  coincides with the no-arbitrage price  $\pi_s$ . The two coincide also at the terminal date.

In general, when  $s < t < T$  the rate-adjusted price is different from the no-arbitrage price. Fixed any  $t$ , a bunch of valuations  $\rho_t^T(s, h_T)$  are available, obtained by solving several problems as (10)

defined on different time intervals  $[s, T]$  with  $s < t$ . However, rate-adjusted prices with different starting points are consistent within them. Indeed, consider  $s_1 \leq s_2 \leq t$  and  $h_T \in L_{s_1}^1(\mathcal{F}_T, F^T)$ . Then, the martingale property of forward prices ensures that

$$\mathbb{E}_{s_1}^{F^T} \left[ e^{r_{s_2}^T(T-t)} \rho_t^T(s_2, h_T) \right] = \mathbb{E}_{s_1}^{F^T} \left[ e^{r_{s_1}^T(T-t)} \rho_t^T(s_1, h_T) \right].$$

In addition, by moving  $s_2$  to  $t$  from the left, if  $r_{s_2}^T$  converges in probability to  $r_t^T$ , we get<sup>9</sup>

$$\rho_t^T(s_2, h_T) \xrightarrow{P} \pi_t(h_T), \quad s_2 \rightarrow t^-.$$

The proper link between  $\rho^T$  and  $\pi$  is given by

$$\rho_t^T(s, h_T) = e^{-(r_s^T - r_t^T)(T-t)} \pi_t(h_T) = e^{r_s^T(t-s)} \frac{\pi_s(1_T)}{\pi_t(1_T)} \pi_t(h_T). \quad (12)$$

This equality can also be read as a parity relation between  $\rho^T$  and  $\pi$ :

$$e^{r_s^T(T-t)} \rho_t^T(s, h_T) = e^{r_t^T(T-t)} \pi_t(h_T),$$

where both sides deliver the price at time  $t$  of a forward contract on  $h_T$  for date  $T$ .

The difference between rate-adjusted and no-arbitrage prices is genuinely due to the term structure of interest rates in the market: in case rates are constant, the distortion between  $\rho^T$  and  $\pi$  disappears. In fact, problem (10) generalizes the dynamics of no-arbitrage prices of problem (8) for constant rates of interest.  $\rho^T$  solves in  $[s, T]$  the analogous differential equation of  $\pi$ , where interest rates are replaced by  $\mathcal{F}_s$ -measurable bond yields and the forward measure replaces the risk-neutral one. In this perspective rate-adjusted prices may be seen as a generalization of no-arbitrage prices in floating-rate markets. Moreover, when pure discount bonds are considered,  $\rho_t^T(s, 1_T)$  is exactly the no-arbitrage price of the zero-coupon bond in a parallel market in which interest rates are fixed and equal to the yield over  $[s, T]$ .

Figure 1 considers U.S. Treasury bonds in January 1987 with increasing expiry  $T$  of 5, 10, 20 and 30 years. We fix  $s = 0$  at January 1987 and consider annual  $t$  up to the redemption date.<sup>10</sup> In particular, the yield curve associated with U.S. Treasury bonds in January 1987 is increasing. In the four graphs of Figure 1 we plot the rate-adjusted prices  $\rho_t^T(0, 1_T)$  of these securities and the ex-post realizations of no-arbitrage prices  $\pi_t(1_T)$ , observed in later years in the market. Treasury bonds rate-adjusted and no-arbitrage prices are indistinguishable at the initial date and near maturity.

<sup>9</sup>This fact can be easily assessed by applying the continuous mapping theorem.

<sup>10</sup>Data are provided by the Federal Reserve Board at daily frequency: see Gürkaynak, Sack, and Wright (2007).

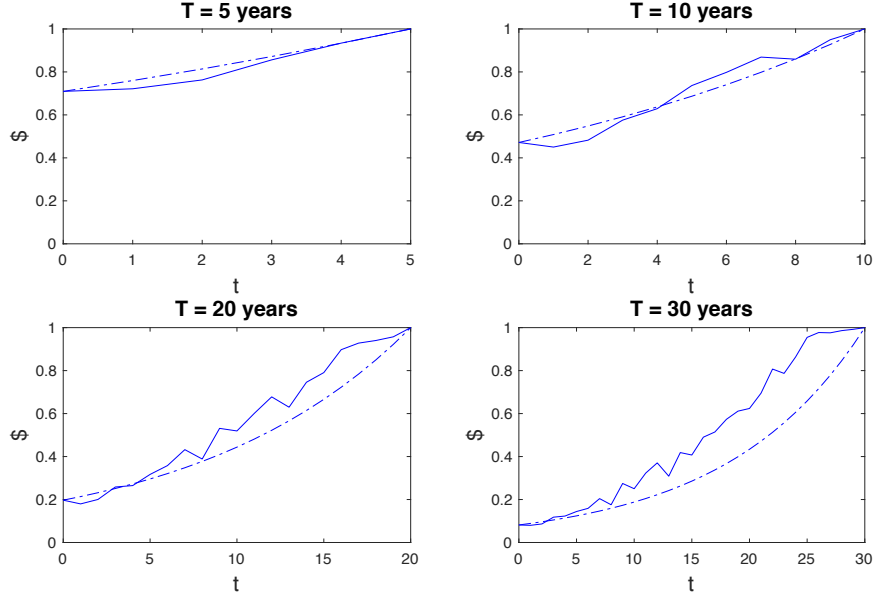


Figure 1: Fix  $s = 0$  at January 1987 and consider increasing maturities  $T$  of U.S. Treasury bonds. Dashed lines depict the annual values of  $\rho_t^T(0, 1_T)$  in  $[0, T]$ . Solid lines represent the (ex-post) realizations of no-arbitrage prices  $\pi_t(1_T)$  at any year  $t$  until expiration.

Different values of  $r_0^T$  may, however, induce an overestimation or underestimation of bond prices, which can be evaluated in future periods according to the actual realizations of bond prices in the market. Specifically,  $\rho_t^T(0, 1_T) < \pi_t(1_T)$  if and only if  $r_0^T > r_t^T$ , a property that holds for any non-negative payoff  $h_T$ , too. The relation between yields at different times determines, indeed, the discrepancy between rate-adjusted and no-arbitrage valuation.

### 3.2 Rate-adjusted prices as indifference prices

Rate-adjusted prices can be interpreted as indifference prices that allow investors to hedge from interest rates variability. We bolster this intuition by describing a simple self-financing strategy in our market.

We consider an investor who incurs an expenditure of  $\pi_s(1_T)$  at time  $s$  in order to buy a self-financing portfolio that delivers some units of a risky security with payoff  $h_T$  at maturity. Specifically, at date  $s$  she buys a zero-coupon bond with expiration  $T$ . At a later time  $t$  she rebalances her allocation, worth  $\pi_t(1_T)$ , by purchasing the amount  $\pi_t(1_T)/\pi_t(h_T)$  of the risky security, which is traded in the market at the no-arbitrage price. Then, she holds this portfolio until maturity.

We suppose that another investor faces the same market. She is less sophisticated than the previous one and pretends that the zero-coupon bond is traded at the fixed rate  $r_s^T$ . Her belief is

consistent with the observation of the  $T$ -bond price at time  $s$ , that is  $\pi_s(1_T) = e^{-r_s^T(T-s)}$ , which coincides with  $\rho_s^T(s, 1_T)$ . Similarly to the first agent, she plans to go long on the pure discount bond at time  $s$  and to entirely liquidate her position at  $t$  in order to purchase all the units of the risky security that she can afford.

Since the second investor disregards the term structure of bond prices, she expects a position worth  $e^{-r_s^T(T-t)}$  at time  $t$  and, consequently, an erroneous amount of the risky security at maturity. However, the two agents would end up with the same number of units of  $h_T$  if the second investor used  $\rho^T$  for pricing the risky security. Indeed,  $\rho_t^T(s, h_T)$  is the theoretical price of  $h_T$  that makes the terminal values of both portfolios coincide.

This description substantiates the hedging nature of  $\rho^T$ . Although the first investor properly exploits the variability of rates, the second one pretends to face a flat term structure. Given an identical initial expenditure, they obtain the same outcome if the less sophisticated agent employs rate-adjusted prices for the valuation of marketed payoffs.

### 3.3 Rate-adjusted prices as conversion prices

We now provide an interpretation of rate-adjusted prices in terms of conversion prices of mandatory convertible bonds with contingent conversion prices. To be in line with the previous notation, we assume that the payoff  $h_T$  represents a stock price and we denote it by  $X_T$ .

We consider a convertible bond issued at time  $s$  with maturity  $T$ , unitary price and fixed rate  $r_s^T$ . At a given time  $t < T$ , the security is compulsorily converted into a proper amount of shares. The conversion price and ratio are determined at time  $t$  by no-arbitrage considerations.

Suppose that at time  $s$  the issuer of the hybrid security purchases  $1/\pi_s(1_T)$  units of a pure discount bond (traded in the market) with redemption date  $T$ . The total cashflow at  $s$  is null. Immediately before the conversion, the value of the position is  $\pi_t(1_T)/\pi_s(1_T)$  while the convertible bond is worth  $e^{r_s^T(t-s)}$ . On the one hand, the holdings of the issuer at time  $t$  correspond to  $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$  shares with price  $\pi_t(X_T)$ . On the other, the bond conversion takes place according to the relation  $e^{r_s^T(t-s)} = p_t q_t$ , where  $p_t$  and  $q_t$  are  $\mathcal{F}_t$ -measurable conversion price and ratio. The absence of arbitrage opportunities implies that  $q_t$  needs to coincide with the amount of shares  $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$ . Otherwise, an initial null expenditure would ensure a positive outcome (in terms of stocks) at terminal date  $T$ . Since  $p_t = e^{r_s^T(t-s)}/q_t$ , we deduce that  $p_t$  takes the expression of eq. (12). As a result,  $p_t$  equals the rate-adjusted price  $\rho_t^T(s, X_T)$ .

The construction of the previous hybrid asset characterizes  $\rho^T$  as a conversion price from bonds to stocks in an arbitrage-free market. The reasoning highlights the ability of  $\rho^T$  to quantify the



risk exposure when financing through fixed-rate securities in a market with stochastic rates where zero-coupon bonds are traded.

### 3.4 Role of the yield $r_s^T$ in the rate-adjusted problem

Problem (10) is not the sole way to generalize the differential relation  $\mathcal{D}\pi = r\pi$  of problem (8). Indeed, under the forward measure, any process  $f \in \mathcal{U}_s^1$  defined by  $f_t = e^{-\xi_s(T-t)}\mathbb{E}_t^{F^T}[h_T]$  with  $\xi_s \in L^0(\mathcal{F}_s)$  solves

$$\begin{cases} \mathcal{D}f_t = \xi_s f_t & t \in [s, T) \\ f_T = h_T \end{cases} \quad (13)$$

where  $h_T \in L_s^1(\mathcal{F}_T, F^T)$ . For instance, assume that the pure discount bond price  $\pi_s(1_T)$  is differentiable with respect to maturity  $T$ . We can set  $\xi_s$  as the instantaneous forward rate with settlement date  $T$  contracted at time  $s$ , i.e.

$$\xi_s = -\frac{\partial \log \pi_s(1_T)}{\partial T} = r_s^T + \frac{\partial r_s^T}{\partial T}(T-s).$$

The associated process  $f$  satisfies the parity relation with the no-arbitrage price  $\pi$ , namely  $e^{\xi_s(T-t)}f_t = e^{r_t^T(T-t)}\pi_t(h_T)$ , and both terms of the equality match the forward price of  $h_T$  at  $t$ . In addition, in case interest rates are constant,  $\xi_s$  collapses to the instantaneous rate and  $f$  specializes to the no-arbitrage price. However, although conceptually attracting, the instantaneous forward rate has an important drawback: at the initial date  $s$ ,  $f_s$  is different from the no-arbitrage price  $\pi_s$  when rates are time-varying.<sup>11</sup>

The initial  $f_s$  equals the no-arbitrage price  $\pi_s$  if and only if  $\xi_s = r_s^T$ . This fact motivates the choice of the yield to maturity  $r_s^T$  in the formalization of problem (10) instead of alternative generalizations. Indeed, the arising solution  $\rho^T$  exactly provides the no-arbitrage price at instant  $s$ . Furthermore,  $\rho^T$  is the unique process – among all possible  $f$  that solve problems as (13) – that features this correctness property.

### 3.5 Long-term relations between rate-adjusted and no-arbitrage prices

We now investigate the relation between  $\rho_t^T$  and  $\pi_t$  when maturity  $T$  increases. Although the yields  $r_s^T$  and  $r_t^T$  converge to the same long-term yield, the asymptotic relations between  $\rho_t^T$  and  $\pi_t$  are

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<sup>11</sup>Another interesting parametrization is provided by  $\xi_s = \mathbb{E}_s[Y_T]$ . This quantity actually coincides with the instantaneous forward rate if the expectation hypothesis of Fisher (1930) holds. The solution  $f$  of problem (13) can be easily determined. As in the previous example,  $\xi_s$  reduces to the instantaneous rate in case rates are constant and, accordingly,  $f$  becomes indistinguishable from  $\pi$ . Nevertheless, when interest rates are floating over time, there is an initial mispricing because  $f_s$  is different from  $\pi_s$ .

delicate. As discussed in Subsection 2.2, we adopt the assumptions of Qin and Linetsky (2017) and we rely, in particular, on Theorem 3.2 therein.

**Proposition 3** *Under the Assumptions 1, if  $h_T \in L_s^1(\mathcal{F}_T, F^T)$ , then for any  $t > s$*

$$\frac{\rho_t^T(s, h_T) - \pi_t(h_T)}{\mathbb{E}_t^{F^T}[h_T]} \xrightarrow{P} 0, \quad T \rightarrow +\infty$$

and, in case  $h_T$  is also strictly positive,

$$\frac{\log \rho_t^T(s, h_T) - \log \pi_t(h_T)}{T - t} \xrightarrow{P} 0, \quad T \rightarrow +\infty.$$

The asymptotic behaviors of Proposition 3 are valid for any choice of  $s$  and  $t > s$ , which are fixed before taking the limit across increasing horizons. However, convergences are taken after a proper rescaling. In the next proposition we focus on the ratio between rate-adjusted and no-arbitrage prices.

**Proposition 4** *Under the Assumptions 1, if  $h_T \in L_s^1(\mathcal{F}_T, F^T)$ , then for all  $t > s$*

$$\mathbb{E}_s^{F^T} \left[ \frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \right] \xrightarrow{P} e^{(r^\infty - r_s^t)(t-s)}, \quad T \rightarrow +\infty$$

and

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, \quad T \rightarrow +\infty.$$

The difference between the long-term yield and the bond yield  $r_s^t$  determines the limit for the expected ratio between  $\rho_t^T$  and  $\pi_t$  under the forward measure. In addition, this ratio turns out to have a well-defined long-run limit in probability. Asymptotically  $\rho_t^T$  differs from  $\pi_t$  by a multiplicative  $\mathcal{F}_t$ -measurable factor associated with the discounted long bond. This factor is actually the transient component in the long-term pricing kernel decomposition recalled in Subsection 2.3.

### 3.6 Rate-adjusted prices as long-term prices

As described by Qin and Linetsky (2017) and recapped in eq. (5), under the Assumptions 1 the long-term growth rate of the pricing kernel  $M_{s,t}$  is the long-term yield  $r^\infty$ . Since  $r^\infty$  is the limit of  $r_s^T$  when  $T$  goes to infinity and  $r_s^T$  is the leading parameter of problem (10), we move  $T$  to infinity in this problem and we analyze the behavior of the solutions over increasing horizons. We start from solving the *long-term* rate-adjusted problem

$$\begin{cases} \mathcal{D}f_t = r^\infty f_t & t \in [s, T) \\ f_T = h_T \end{cases} \quad (14)$$

with  $h_T \in L_s^1(\mathcal{F}_T, F^\infty)$ . Differently from problem (10), here the long-term yield replaces  $r_s^T$  and the long-term forward measure is employed.

**Theorem 5** *There exists a unique solution of problem (14) in  $\mathcal{U}_s^1$ , given by*

$$\rho_t^\infty(s, h_T) = e^{-r^\infty(T-t)} \mathbb{E}_t^{F^\infty} [h_T] \quad \forall t \in [s, T]. \quad (15)$$

We refer to  $\rho_t^\infty(s, h_T)$  as the *long-term rate-adjusted price* of  $h_T$  at time  $t$  in the interval  $[s, T]$ .

Problem (14) and problem (10) differ in two main aspects. The first one is the horizon of the forward measure, which is either infinite or finite. The second one is the randomness of the multiplier. Indeed,  $r^\infty$  is a number while  $r_s^T$  is properly an  $\mathcal{F}_s$ -measurable random variable.

After solving problems (10) and (14) separately, we address the issue of the convergence of  $\rho^T$  to  $\rho^\infty$  when the horizon increases. In particular, we solve a sequence of differential problems related to a term structure of horizons, whose solutions tend to the solution of the long-term rate-adjusted problem.

In order to do so, we disentangle the instant at which the payoff under scrutiny is paid from the horizon. We assume that the maturity of the security under consideration is  $\tau$ , while the horizon is  $T \geq \tau$ . The rate-adjusted pricing problem of Subsection 3.1 can, then, be rewritten as

$$\begin{cases} \mathcal{D}f_t = r_s^T f_t & t \in [s, \tau) \\ f_\tau = h_\tau \end{cases} \quad (16)$$

with  $h_\tau \in L_s^1(\mathcal{F}_\tau, F^T)$ . If  $\tau = T$ , we precisely retrieve problem (10). In the more general formulation considered here, the unique solution in  $\mathcal{U}_s^1$  over  $[s, \tau]$  is

$$\rho_t^T(s, h_\tau) = e^{-r_s^T(\tau-t)} \mathbb{E}_t^{F^T} [h_\tau] \quad (17)$$

for all  $t \in [s, \tau]$ .<sup>12</sup> Finally, we investigate the convergence of  $\rho_t^T(s, h_\tau)$  when  $T$  increases. Under mild assumptions, the solution of problem (16) converges to the long-term rate-adjusted price that solves problem (14).

**Proposition 6** *Under the Assumptions 1, suppose that  $h_\tau \in L^1(\mathcal{F}_\tau, F^T)$  for all  $T \geq \tau$  and  $G_{t,\tau}^T h_\tau$  is convergent in  $L^1(P)$  when  $T$  goes to infinity for all  $t \in [s, \tau]$ . Then, for all  $t \in [s, \tau]$ ,*

$$\rho_t^T(s, h_\tau) \xrightarrow{P} \rho_t^\infty(s, h_\tau), \quad T \rightarrow +\infty.$$

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<sup>12</sup>This rate-adjusted price is fundamental for the valuation of cashflows that we illustrate in the next subsection.

As a result,  $\rho^\infty$  can be properly interpreted as a rate-adjusted price for the long run. Indeed, the asset valuation through  $\rho^\infty$  exploits  $r^\infty$ , which constitutes the stochastic discount factor long-term growth rate.

### 3.7 Valuation of cashflows

Eq. (17) describes the rate-adjusted price of a payoff  $h_\tau$  in the time window  $[s, T]$  with  $s \leq \tau \leq T$ . Disentangling the payment date  $\tau$  from the horizon  $T$  allows us to easily provide the rate-adjusted price for payoff streams. First, denote by  $\alpha_t^T(s, h_\tau)$  the ratio between the rate-adjusted price and the no-arbitrage price of  $h_\tau$  at any time  $t \in [s, \tau]$ :

$$\rho_t^T(s, h_\tau) = \alpha_t^T(s, h_\tau) \pi_t(h_\tau) = e^{-(r_s^T - r_t^T)(\tau - t)} \frac{\mathbb{E}_t^{F^T}[h_\tau]}{\mathbb{E}_t^{F^\tau}[h_\tau]} \pi_t(h_\tau).$$

Note that  $\alpha_t^T(s, h_T)$  does not depend on  $h_T$  and it is consistent with eq. (12).

Consider now a cashflow  $h = \{h_{\tau_i}\}_{i=1}^N$  with  $N \in \mathbb{N}$ ,  $s \leq \tau_1 < \dots < \tau_N \leq T$  and  $h_{\tau_i} \in L^1(\mathcal{F}_{\tau_i}, F^T)$  for all  $i$ . The rate-adjusted price  $\rho_t^T(s, h)$  of the cashflow follows from standard linear pricing and the related rate adjustment is the ratio between  $\rho_t^T(s, h)$  and the no-arbitrage price  $\pi_t(h)$ :

$$\rho_t^T(s, h) = \sum_{\tau_i \geq t} \rho_t^T(s, h_{\tau_i}) = \alpha_t^T(s, h) \pi_t(h).$$

Specifically,

$$\alpha_t^T(s, h) = \frac{\sum_{\tau_i \geq t} \alpha_t^T(s, h_{\tau_i}) \pi_t(h_{\tau_i})}{\sum_{\tau_i \geq t} \pi_t(h_{\tau_i})}. \quad (18)$$

The evolution of  $\alpha^T$ , as well as the difference between  $\rho^T$  and  $\pi$ , can be useful for quantifying the interest rate risk exposure of the cashflow over time. Indeed, the no-arbitrage price of  $h$  decomposes into the product of  $\rho^T$  and the inverse of  $\alpha^T$ . Since the rate-adjusted price captures the long-term interest rate risk of  $h$  (as we further illustrate in Subsection 4.2), the short-term interest rate exposure flows into the adjustment. This approach is fruitful for the risk management of securities that feature long maturities and sensitivity to shocks in the term structure of rates as interest rate derivatives, life insurances and annuities (see Section 5).<sup>13</sup>

<sup>13</sup>In addition, in a buy-and-hold portfolio strategy, the evolution of  $\alpha^T$  can provide an indication of the interest rate risk to bear in case the position is unexpectedly closed before maturity.

## 4 Pricing kernel growth

From the definition of problem (10) it is clear that  $\mathcal{D}\rho^T$  is weakly time-differentiable in  $[s, T]$ . The same also holds for weak time-derivatives of higher orders. As a result, the rate-adjusted price  $\rho^T$  is infinitely weakly time-differentiable and so it belongs to  $\mathcal{U}_s^\infty$ . A parallel reasoning ensures the infinite weak differentiability of  $\rho^\infty$ .

Since  $\mathcal{D}(\xi_s u) = \xi_s \mathcal{D}u$  for all  $u \in \mathcal{U}_s^\infty$  and  $\xi_s \in L^0(\mathcal{F}_s)$ , the weak time-derivative in  $[s, T]$  defines an  $L^0$ -linear operator  $\mathcal{D} : \mathcal{U}_s^\infty \rightarrow \mathcal{U}_s^\infty$  and  $\rho^T$  satisfies the eigenvalue-eigenvector problem

$$\mathcal{D}\rho^T = r_s^T \rho^T, \quad (19)$$

where the eigenvalue belongs to  $L^0(\mathcal{F}_s)$ . Accordingly,  $\rho^\infty$  solves

$$\mathcal{D}\rho^\infty = r^\infty \rho^\infty \quad (20)$$

where the eigenvalue is a positive number.

As in Hansen and Scheinkman (2009), we suppose that the payoff  $h_T$  is positive so that  $\rho^T$  and  $\rho^\infty$  are positive. Hence,  $\rho^T$  and  $\rho^\infty$  are *principal eigenvectors* related to  $r_s^T$  and  $r^\infty$  respectively. This property is key to the economic theory because it is in line with the Perron-Frobenius theory, usually employed in Markovian environments and successfully applied by Ross (2015) and Qin and Linetsky (2016). Indeed, when rates are constant, the principal eigenvalue associated with a differential price operator turns out to be the growth rate of the stochastic discount factor (Hansen and Scheinkman, 2009).

In this section we analyze the relation between the eigenvalues of several differential pricing problems and pricing kernel growth rates in our stochastic-rate market. To build our theory, we introduce the notion of rate-adjusted pricing kernel. Proofs can be found in Appendix C.

### 4.1 Finite- and infinite-horizon pricing kernel decomposition

The long-term eigenvalue problem of eq. (20) actually exploits the pricing kernel long-term growth rate, that is the long-term yield. Indeed, as shown by Qin and Linetsky (2017), under the Assumptions 1  $M_{s,t}$  satisfies the long-run decomposition of eq. (5), i.e.

$$M_{s,t} = e^{-r^\infty(t-s)} \frac{b_s^\infty}{b_t^\infty} G_{s,t}^\infty.$$

The question, here, is whether an analogous property holds for the  $T$ -horizon problem of eq.

(19). We need first to characterize the finite-horizon pricing kernel growth rate.

As introduced in Subsection 2.3, the pricing kernel  $M_{s,t}$  satisfies

$$M_{s,t} = \frac{e^{-r_s^T(T-s)}}{e^{-r_t^T(T-t)}} G_{s,t}^T = \frac{e^{r_s^T s} \pi_s(1_T)}{\pi_s(1_{T-t})} \frac{e^{-r_s^T s} \pi_s(1_{T-t})}{\pi_t(1_T)} G_{s,t}^T$$

for all  $T > t + s$ . In the next proposition we assess the asymptotic behaviors of the last three factors by using Theorem 3.2 in Qin and Linetsky (2017). Note that the ratio  $\pi_s(1_T)/\pi_s(1_{T-t})$  is the return on a forward rate agreement between  $T - t$  and  $T$  contracted at date  $s$ .

**Proposition 7** *Under the Assumptions 1, for all  $s > 0$  and  $t > s$*

$$\begin{aligned} \frac{e^{r_s^T s} \pi_s(1_T)}{\pi_s(1_{T-t})} &\xrightarrow{P} e^{-r^\infty(t-s)}, & T \rightarrow +\infty \\ \frac{e^{-r_s^T s} \pi_s(1_{T-t})}{\pi_t(1_T)} &\xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, & T \rightarrow +\infty \\ G_{s,t}^T &\xrightarrow{P} G_{s,t}^\infty, & T \rightarrow +\infty. \end{aligned}$$

The first convergence shows that, in addition to being a limit of bond yields,  $r^\infty$  is also a limit of continuously compounded forward rates contracted at  $s$ , as the convergence of  $\pi_s(1_T)/\pi_s(1_{T-t})$  suggests.<sup>14</sup> The second convergence involves the transient component of  $M_{s,t}$ . The limit of this factor is captured by the ratio of the discounted values of the long bond. Finally, the third convergence regards the permanent (or martingale) component of the pricing kernel. At any finite horizon, this term consists of the Radon-Nikodym density of the  $T$ -forward measure. When the horizon is infinite, the long-term forward measure appears.

Consistently with the three simultaneous convergences of Proposition 7, when the horizon is finite we call growth term of  $M_{s,t}$  the first ratio in the proposition, i.e.

$$\frac{e^{r_s^T s} \pi_s(1_T)}{\pi_s(1_{T-t})} = e^{-r_s^T(T-2s) + r_s^T t(T-t-s)}. \quad (21)$$

Hence,  $M_{s,t}$  features a random growth term in  $L^0(\mathcal{F}_s)$  determined by bond yields computed at time  $s$ . In case interest rates are constant, this quantity actually reduces to  $e^{-r(t-s)}$ , which captures the growth of the pricing kernel  $e^{-r(t-s)} L_{s,t}$ .

As it is apparent from eq. (21), the growth rate of  $M_{s,t}$  is not exactly  $r_s^T$  and so, differently from the long-term yield, it does not agree with the eigenvalue problem  $\mathcal{D}\rho^T = r_s^T \rho^T$ . Therefore,

<sup>14</sup>This approach is reminiscent of the construction of Backus, Gregory, and Zin (1989) and of the methodology of Alvarez and Jermann (2005), who highlight the informativeness of prices of bonds with distant maturities on pricing kernel persistence.

in order to isolate the growth rate  $r_s^T$  in finite horizons, we introduce the notion of *rate-adjusted pricing kernel*, which parallels the one of rate-adjusted prices.

## 4.2 Rate-adjusted pricing kernel

Given any payoff  $h_T$ , we can write no-arbitrage and rate-adjusted prices at time  $t$  as

$$\pi_\tau(h_T) = \mathbb{E}_\tau[M_{\tau,T}h_T], \quad \rho_\tau^T(s, h_T) = \mathbb{E}_\tau[N_{\tau,T}^T h_T],$$

where, for all  $\tau, t$  in  $[s, T]$  with  $\tau \leq t$ ,

$$M_{\tau,t} = \frac{e^{r_t^T(T-t)}}{e^{r_\tau^T(T-\tau)}} G_{\tau,t}^T, \quad N_{\tau,t}^T = \frac{e^{r_s^T(T-t)}}{e^{r_s^T(T-\tau)}} G_{\tau,t}^T.$$

We call  $N_{s,t}^T$  the *rate-adjusted pricing kernel*. In particular,

$$N_{s,t}^T = e^{(r_s^T - r_t^T)(T-t)} M_{s,t}$$

and it coincides with  $M_{s,t}$  when rates are constant. The adjustment coefficient between  $N_{s,t}^T$  and  $M_{s,t}$  is the inverse of the one between  $\pi$  and  $\rho^T$  pointed out in eq. (12). In addition, the expected values of  $M_{s,t}$  and  $N_{s,t}^T$  differ only in the yields  $r_s^t$  and  $r_s^T$ :

$$\mathbb{E}_s[M_{s,t}] = e^{-r_s^t(t-s)}, \quad \mathbb{E}_s[N_{s,t}^T] = e^{-r_s^T(t-s)}.$$

Since  $N_{s,t}^T = e^{-r_s^T(t-s)} G_{s,t}^T$ , its growth rate is  $r_s^T$ , in agreement with the rate-adjusted eigenvalue problem  $\mathcal{D}\rho^T = r_s^T \rho^T$ . Moreover, differently from  $M_{s,t}$ , the rate-adjusted pricing kernel is explicitly dependent on the horizon  $T$  under scrutiny. In the next proposition we establish the convergence of  $N_{s,t}^T$  when  $T$  becomes arbitrarily large.

**Proposition 8** *Under the Assumptions 1, for all  $s > 0$  and  $t > s$ ,*

$$N_{s,t}^T \xrightarrow{P} e^{-r^\infty(t-s)} G_{s,t}^\infty, \quad T \rightarrow +\infty.$$

Therefore, we define the *long-term rate-adjusted pricing kernel* by

$$N_{s,t}^\infty = e^{-r^\infty(t-s)} G_{s,t}^\infty \tag{22}$$

and  $r^\infty$  naturally arises as growth rate for  $N_{s,t}^\infty$ .

A formalization of the fact that  $r_s^T$  and  $r^\infty$  are the rate-adjusted pricing kernel growth rates for

finite (or infinite) horizons can be easily obtained in differential terms. When the horizon is finite, we can consider the problem

$$\begin{cases} \mathcal{D}f_t = -r_s^T f_t & t \in [s, T] \\ f_s = 1 \end{cases} \quad (23)$$

under the physical measure. When the horizon is infinite, we analyze (still under  $P$ )<sup>15</sup>

$$\begin{cases} \mathcal{D}f_t = -r^\infty f_t & t \in [s, +\infty) \\ f_s = 1. \end{cases} \quad (24)$$

**Theorem 9** *Under the measure  $P$ ,  $\{N_{s,t}^T\}_t$  solves problem (23) in  $\mathcal{U}_s^1$  with finite  $T$  and  $\{N_{s,t}^\infty\}_t$  solves problem (24) in  $\mathcal{U}_s^1$  with infinite  $T$ .*

The last theorem formalizes the growth terms  $r_s^T$  and  $r^\infty$  for the rate-adjusted pricing kernel at finite and infinite horizons. The results are consistent with the eigenvalue problems  $\mathcal{D}\rho^T = r_s^T \rho^T$  and  $\mathcal{D}\rho^\infty = r^\infty \rho^\infty$  satisfied by rate-adjusted prices.

Problems (23) and (24) generalize the differential relation  $\mathcal{D}M = -rM$  which is satisfied by the pricing kernel when interest rates are constantly equal to  $r$ . When rates are stochastic, in standard diffusive models, the pricing kernel follows the dynamics

$$dM_{s,t} = -Y_t M_{s,t} dt - \nu_t M_{s,t} dW_t^P,$$

where  $\nu_t$  is the market price of risk (see, for instance, Subsection 6.2.2). The stochastic rate is present in the drift of the pricing kernel. However, any  $Y_t$  alone is not able to capture the growth rate of  $M_{s,t}$  and it is not known ex ante. This last feature forbids any eigenvalue-eigenvector formulation of the problem of identifying a growth rate for  $M_{s,t}$ .

Problems (23) and (24) are not satisfied by the pricing kernel  $M_{s,t}$  when rates of interest are stochastic as well as no-arbitrage prices do not solve problems (10) and (14). The deep reason for these failures can be understood through the lens of the Hansen-Scheinkman decomposition.

By comparing eq. (22) with the long-term decomposition of the pricing kernel in eq. (5), it is apparent that  $N_{s,t}^\infty$  features the same growth rate  $r^\infty$  of  $M_{s,t}$ , as well as the same martingale component  $G_{s,t}^\infty$ . Nevertheless, the transitory component of  $N_{s,t}^\infty$  is deterministic and equal to 1. Therefore, employing rate-adjusted prices for asset valuation means using a stochastic discount factor that is free from transitory effects in its long-term Hansen-Scheinkman decomposition. See

<sup>15</sup>The last problem uses weak time-derivatives in  $[s, +\infty)$  that can be defined by replacing  $T$  with  $+\infty$  in Definition 1. The module  $\mathcal{U}_s$  modifies accordingly by omitting the left-continuity of processes  $u$  at  $T$  but additionally requiring that  $\int_s^{+\infty} \mathbb{E}_s[|u_\tau|] d\tau$  belongs to  $L^0(\mathcal{F}_s)$ .



also the previous eq. (6) that highlights the role of the transient component of  $M_{s,t}$  in the price of any marketed payoff.

When interest rates are constant, the transient component of the stochastic discount factor is 1 and so the difference between  $M_{s,t}$  and  $N_{s,t}^\infty$  vanishes. On the contrary, when rates are stochastic, a trivial temporary term can be retrieved only in  $N_{s,t}^\infty$ . For this reason, rate-adjusted prices allow us to generalize both price and stochastic discount factor dynamics from the constant-rate case. Rate-adjusted prices aggregate infinitesimal randomness to long-run risk exposure because  $N_{s,t}^T$  is free from any transitory component. Moreover, bond yields are the proper financial variables that translate local riskiness to long-run risks through rate-adjusted prices. We summarize our main findings in Table 1.

Table 1: **Summary of results.** Comparison of numéraires, pricing relations and pricing kernel growth rates between constant and stochastic rates. Finite and infinite horizons are considered. In the stochastic case, rate-adjusted prices and pricing kernels are considered.

	Constant rate	Stochastic rates	
		finite $T$	infinite $T$
Numéraire	money market	zero-coupon bond	long bond
Measure	$Q$	$F^T$	$F^\infty$
Yield	$r$	$r_s^T$	$r^\infty$
Price	$\pi$	$\rho^T$	$\rho^\infty$
Return-rate relation	$\mathcal{D}\pi = r\pi$	$\mathcal{D}\rho^T = r_s^T \rho^T$	$\mathcal{D}\rho^\infty = r^\infty \rho^\infty$
Pricing kernel growth rate	$r$	$r_s^T$	$r^\infty$

## 5 Applications of rate-adjusted pricing

The introduction of rate adjustments is fruitful in several areas of finance where managing the risk of long-lasting cashflows is important. To illustrate the ability of rate-adjusted pricing at disentangling long- and short-term interest risk, we consider examples coming from derivative pricing, life insurances, mortgages and pension funds.

### 5.1 Interest rate swaps

Consider an interest rate swap contract with periodic payment exchanges at dates  $\tau_1, \dots, \tau_N$  in the time interval  $[0, T]$  with  $\tau_N = T$ . The reference period for the first payment is  $[\tau_0, \tau_1]$  with  $0 < \tau_0 < \tau_1$ . At any payment date  $\tau_i$ , two parties exchange a fixed leg  $k$  with a floating leg made by the LIBOR rate  $\mathcal{L}(\tau_{i-1}, \tau_i)$ , which satisfies  $(1 + (\tau_i - \tau_{i-1})\mathcal{L}(\tau_{i-1}, \tau_i))\pi_{\tau_{i-1}}(1_\tau) = 1$ . Hence, the

cashflow  $h$  of the swap payer is given by  $h_{\tau_i} = (\mathcal{L}(\tau_{i-1}, \tau_i) - k)(\tau_i - \tau_{i-1})$  for any  $i = 1, \dots, N$ . Then, the no-arbitrage price of any  $h_{\tau_i}$  at time  $t \in [0, \tau_i]$  is

$$\pi_t(h_{\tau_i}) = \begin{cases} \pi_t(1_{\tau_{i-1}}) - \pi_t(1_{\tau_i}) - k(\tau_i - \tau_{i-1})\pi_t(1_{\tau_i}) & t \in [0, \tau_{i-1}] \\ h_{\tau_i}\pi_t(1_{\tau_i}) & t \in [\tau_{i-1}, \tau_i] \end{cases}$$

and the premium  $k$  is set in a way that  $\pi_0(h) = 0$ . We compute the no-arbitrage and the rate-adjusted price of the swap by following the linearity approach of Subsection 3.7.

We simulate several cashflows of the swap by assuming a Vasicek (1977) term structure of rates as in Subsection 6.1.1. We fix the first payment date  $\tau_1$  and the maturity  $T$  but we increase the number of exchanges in between. In this way we show that the rate-adjusted price relies mainly on the *maturity* of the swap, while the rate adjustment depends on the *tenor*. Indeed, short tenors are associated with a large short-term interest rate risk, which is much less relevant when payment exchanges are infrequent.

We plot in the left panel of Figure 2 the difference between  $\rho^T$  and  $\pi$  of the swap contracts for decreasing tenors of 48, 24, 12 and 6 months, after setting  $\tau_0 = 12$  and  $\tau_N = 60$  months. By retrieving the intuition of Subsection 3.2, discounting with a constant rate  $r_0^T$  makes a relatively small discrepancy from no-arbitrage pricing when a single payment at  $T$  is considered. However, the difference between the no-arbitrage and the rate-adjusted price of the swap increases when the tenor shortens. Indeed, short tenors are sensitive to temporary interest rate variations and this sensitivity is captured by the rate adjustment.

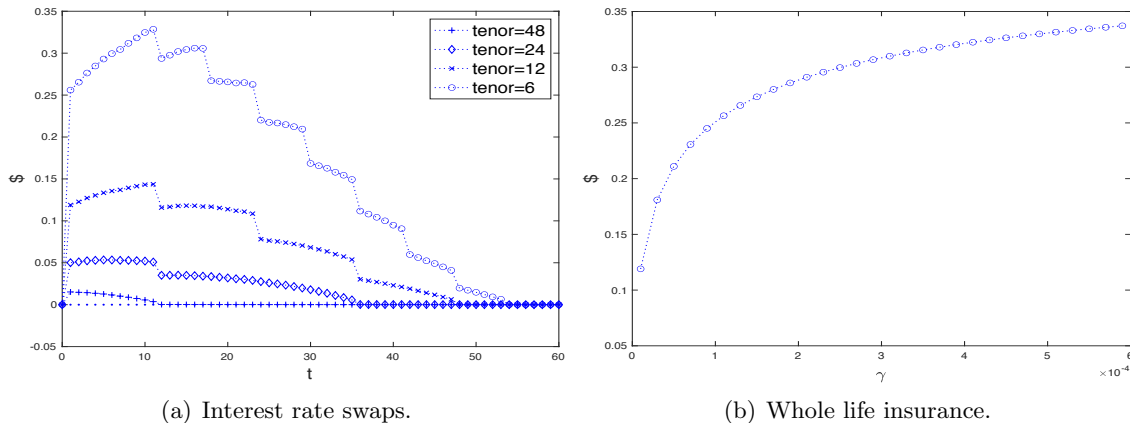


Figure 2: Left panel: difference between  $\rho_t^T(0, h)$  and  $\pi_t(h)$  of the interest rate swaps over time. We set  $\tau_0 = 12$  and  $\tau_N = 60$  months. Decreasing tenors of 48, 24, 12 and 6 months are considered. Right panel: difference between  $\pi_0(h)$  and  $\rho_0^\infty(0, h)$  of the whole life insurance. The horizontal axis contains the parameter  $\gamma$ , which increases with the insured's age at subscription. In both panels a Vasicek term structure of rates is assumed on monthly basis.

## 5.2 Whole life insurance actuarial present value

Consider a whole life insurance that pays a benefit equal to 1 when the subscriber dies. The contract has no preset horizon and the date of the unique payment is unknown *ex ante*. This date is related to the time-until-death, a random variable with probability density function  $\wp$  on  $[0, +\infty)$ , that represents the difference between the insured's age at death and her age at the subscription. Subscribers with different ages feature different future lifetime, and so different  $\wp$ . See Chapters 3 and 4 in Bower, Gerber, Hickman, Jones, and Nesbitt (1997).

The actuarial present value of the insurance is the expectation of the present value of the payment. This value summarizes the overall exposure of the insurance company to the obligation of the contract. Since the horizon of the insurance policy is potentially very large, the actuarial present value depends on the forecast term structure of rates. See, for instance, Panjer and Bellhouse (1980) for the use of stochastic rates in the valuation of life contingencies. The introduction of rate adjustments additionally permits to disentangle the short- and long-term interest rate risk beared by the insurance company.

Suppose that the physical measure (used by the insurance company) coincides with the risk-neutral measure. In this case, in line with the previous notation, we can denote the actuarial present value by  $\pi_0(h)$  and we can consider the *rate-adjusted* actuarial present value  $\rho_0^\infty(0, h)$ . Here  $h$  represents the unitary random payment of the whole life insurance (the subscriber's age is given). Then,

$$\pi_0(h) = \int_0^{+\infty} e^{-r_0^\tau \tau} \wp(\tau) d\tau, \quad \rho_0^\infty(0, h) = \int_0^{+\infty} e^{-r^\infty \tau} \wp(\tau) d\tau.$$

We assume that the time-until-death has cumulative distribution function  $\mathcal{P}(\tau) = 1 - e^{-\gamma \tau^3}$  and in the simulations we move the parameter  $\gamma$  from 0.00001 to 0.00059. The distribution is unimodal and increasing  $\gamma$  means moving the peak of time-until-death towards the date of the insurance subscription. In other words, it means considering shorter future lifetimes or, equivalently, increasing the subscriber's age. The used monthly time grid involves 900 months after the subscription for all  $\gamma$ . Interest rates are simulated following Vasicek (1977) with the specifications described in Subsection 6.1.1.

The right panel of Figure 2 represents the difference between  $\pi_0$  and  $\rho_0^\infty$  when  $\gamma$  increases. The difference is small when  $\gamma$  is low and becomes sizable for high values of the parameter. The rate-adjusted actuarial present value differs more from the actuarial present value when old subscribers (high  $\gamma$ ) are considered. Their future lifetime is short and so short-term rates have an important role in the computation of their actuarial present value. This valuation contrasts with the rate-adjusted

one, which responds substantially to long-term interest rate shocks. The difference between the two actuarial values captures the short-term interest rate risk. On the contrary,  $\rho_0^\infty$  is more similar to  $\pi_0$  for young insured (low  $\gamma$ ) because their time-until-death is large and so their actuarial present value exploits heavily the long-run rates. Hence, considering the rate adjustment provides insights on the short and long-term interest rate risk exposure of the insurance company designing the contract.<sup>16</sup>

### 5.3 Mortgage renegotiation

Consider a fixed-rate mortgage that features periodic payments equal to  $c$  at dates  $\tau_1, \dots, \tau_N$  in the time interval  $[0, T]$  with  $\tau_N = T$ . In line with our notation, we denote by  $h$  the stream of payments  $c$  in the annuity. The amount of money demanded by the borrower is  $v_0$  and the bank imposes the rate  $\xi$ , so that  $v_0 = \sum_{i=1}^N e^{-\xi\tau_i} c$ .

At time  $t > 0$  the borrower wants to assess whether a renegotiation of the loan is convenient. According to the term structure of rates at  $t$ , the market value of the remaining sequence of payments is  $\pi_t(h) = \sum_{\tau_i > t}^T e^{-r_t^{\tau_i}(\tau_i - t)} c$  while the present value obtained from the fixed-rate discounting is  $v_t = \sum_{\tau_i > t}^T e^{-\xi(\tau_i - t)} c$ . Then, the borrower is willing to ask for a renegotiation if  $v_t > \pi_t$ .

If  $\xi = r_0^T$ , then  $v_t$  turns out to be the rate-adjusted price of the remaining payments in the annuity, i.e.  $v_t = \rho_t^T(0, h)$ , and the renegotiation is profitable when the rate-adjusted price exceeds the no-arbitrage price  $\pi_t(h)$ . This requirement boils down to a condition for the rate adjustment of eq. (18), namely  $\alpha_t^T(0, h) > 1$ , where  $\alpha_t^T(0, h)$  equals the ratio  $\sum_{\tau_i > t}^T e^{-r_0^T(\tau_i - t)} / \sum_{\tau_i > t}^T e^{-r_t^{\tau_i}(\tau_i - t)}$ . In general, if  $\xi$  is arbitrary, the latter ratio rewrites with  $\xi$  instead of  $r_0^T$ .

In a nutshell, the mortgage renegotiation rests on the relation between the current term structure of rates and the preset fixed rate. If the latter is equal to  $r_0^T$ , the trade-off between floating and fixed rates is captured exactly by the adjustment between no-arbitrage and rate-adjusted prices.

### 5.4 Pension fund withdrawal

Consider a worker that contributes to a pension plan by regularly devolving a given amount since date  $s$ . The fund guarantees a constant interest rate equal to  $\xi_s$ . Under some circumstances the worker is allowed to withdraw the total amount of her (capitalized) payments. She considers this possibility at time  $t > s$  when her contribution to the fund amounts to  $q_t$ . The value of  $q_t$  that the

<sup>16</sup>International financial regulations as the European Solvency II Directive, require to quantify the overall risk exposure of insurance companies after assuming precise dynamics of rates. Such dynamics depend essentially on the long-term yield (the Ultimate Forward Rate) dictated by European authorities every year: see the Smith and Wilson (2001) procedure described in Jørgensen (2018). The introduction of rate adjustments builds upon this approach by providing a further tool to assess long- and short-term risk exposure of the insurance company under consideration.

fund would guarantee at  $T$  is  $e^{\xi_s(T-t)}q_t$ . On the contrary, according to the current yield curve, the future value of  $q_t$  at  $T$  would be  $q_T = e^{r_t^T(T-t)}q_t$ . As a result, the withdrawal is profitable when  $e^{r_t^T(T-t)} > e^{\xi_s(T-t)}$ . If  $\xi_s = r_s^T$ , this condition translates into a rate adjustment  $\alpha_t^T(s, q_T)$  exceeding 1. Indeed, the comparison of the future values of  $q_t$  rests on the different discounting related to constant-rate and stochastic-rate valuations. The rate adjustment captures the trade-off between the two discounting ways when the constant rate is exactly  $r_s^T$ .

## 6 Single-factor affine interest rate models

We provide an illustration of our theory in arbitrage-free markets with diffusive short-term rates. After specializing to single-factor affine term structure models, we illustrate the differences between rate-adjusted and no-arbitrage prices of pure discount bonds and bond European call options. We also devote some attention to the Feynman-Kac partial differential equations satisfied by the two prices. We finally highlight the dynamics of the pricing kernels (risk-adjusted or not) and their growth terms. Subsection 6.1 considers a fixed-income market while Subsection 6.2 involves a market with both stocks and zero-coupon bonds.

### 6.1 Pricing in a fixed-income market

We start comparing the drifts of zero-coupon bond prices under the measures  $P$ ,  $Q$  and  $F^T$  in a fixed-income market. Then, we make a comparison with rate-adjusted prices. Indeed, when the processes under scrutiny are weakly time-differentiable, the drift is the weak time-derivative (Marinacci and Severino, 2018) and Theorem 2 applies.<sup>17</sup>

We assume that instantaneous rates follow the diffusion process  $dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t^P$  in the time window  $[s, T]$ . Here,  $\mu$  and  $\sigma$  are measurable functions of  $t$  and  $Y_t$ , and  $W_t^P$  denotes a Wiener process under the physical measure. The pure discount  $T$ -bond price at time  $t$  is function of  $t$  and  $Y_t$ . Therefore, Itô's formula permits to determine drift and diffusion coefficient of  $\pi_t(1_T)$ , so that we can write

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = \tilde{\mu}(t, Y_t)dt + \tilde{\sigma}(t, Y_t)dW_t^P.$$

We indicate by  $\nu$  the market price of risk process  $\nu_t = (\tilde{\mu}(t, Y_t) - Y_t)/\tilde{\sigma}(t, Y_t)$ . By Girsanov's theorem, we build a Wiener process under the risk-neutral measure  $Q$  through the stochastic differential  $dW_t^Q = dW_t^P + \nu_t dt$ . The measure  $Q$  triggers a drift change in the evolution of bond

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<sup>17</sup>We use Itô's formula extensively, implicitly postulating that the processes under consideration are continuously differentiable with respect to time and twice continuously differentiable with respect to the spatial variable.

prices:

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = Y_t dt + \tilde{\sigma}(t, Y_t) dW_t^Q.$$

Under  $Q$  the drift coefficient of no-arbitrage bond prices coincides with the instantaneous rate. When changing the numéraire to the pure discount  $T$ -bond, the dynamics of  $\pi_t(1_T)$  under the forward measure become

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = (Y_t + \tilde{\sigma}^2(t, Y_t)) dt + \tilde{\sigma}(t, Y_t) dW_t^{F^T}, \quad (25)$$

where  $W_t^{F^T}$  is a Wiener process under  $F^T$  satisfying  $dW_t^{F^T} = dW_t^Q - \tilde{\sigma}(t, Y_t)dt$ . See Chapter 3 of Brigo and Mercurio (2006). In eq. (25) the drift change is apparent. Differently from the  $Q$ -dynamics, the drift under  $F^T$  depends on the diffusion coefficient of the bond price itself. This feature is not present in  $T$ -bond rate-adjusted prices because their differential satisfies

$$\frac{d\rho_t^T(s, 1_T)}{\rho_t^T(s, 1_T)} = r_s^T dt \quad (26)$$

for any measure under consideration. In agreement with Theorem 2, the weak time-derivative in  $[s, T]$  of  $\rho_t^T(s, 1_T)$  is  $r_s^T \rho_t^T(s, 1_T)$ , while under  $F^T$  the candidate weak time-derivative of  $\pi_t(1_T)$  is  $(Y_t + \tilde{\sigma}^2(t, Y_t))\pi_t(1_T)$ . Since  $Y_t + \tilde{\sigma}^2(t, Y_t)$  is floating over time, under  $F^T$  it is unlikely to find an eigenvalue-eigenvector formulation as problem (10) by using no-arbitrage prices when rates of interest are stochastic. On the contrary, rate-adjusted prices have much simpler dynamics.

### 6.1.1 Zero-coupon bonds in Vasicek (1977)

In single-factor affine term structure models – as Vasicek (1977), Cox, Ingersoll, and Ross (1985) and others – the price of a zero-coupon bond with expiration  $T$  depends on the instantaneous rate  $Y_t$  through the exponential relation  $\pi_t(1_T) = e^{A(t,T) - B(t,T)Y_t}$ , where  $A$  and  $B$  are deterministic functions. Accordingly, the yield to maturity is affine in  $Y_t$ . By Itô's differential rule, the diffusion coefficient of the no-arbitrage price  $\pi_t(1_T)$  is  $\tilde{\sigma}(t, Y_t) = -B(t, T)\sigma(t, Y_t)$  and so, under the forward measure, its drift depends on the volatility of the underlying instantaneous rate:

$$\frac{d\pi_t(1_T)}{\pi_t(1_T)} = (Y_t + B^2(t, T)\sigma^2(t, Y_t)) dt - B(t, T)\sigma(t, Y_t) dW_t^{F^T}.$$

However, the dynamics of the related rate-adjusted price is still the one of eq. (26).

As an example, we consider the Vasicek (1977) model, in which the coefficients of the instantaneous rate are  $\mu(t, Y_t) = k\theta - (k - \sigma\xi)Y_t$  and  $\sigma(t, Y_t) = \sigma$ , where  $k, \theta, \sigma, \xi > 0$  and the market price

of risk is  $\xi Y_t$ . Thus, the evolution of  $Y_t$  under  $Q$  is given by

$$dY_t = k(\theta - Y_t)dt + \sigma dW_t^Q. \quad (27)$$

The short-term rate is mean-reverting towards the value  $\theta$  in the long run at a speed dictated by  $k$ . In addition, volatility is constant over time. Under this specification,

$$A(t, T) = \left( \theta - \frac{\sigma^2}{2k^2} \right) (B(t, T) - T + t) - \frac{\sigma^2}{4k} B^2(t, T)$$

and  $B(t, T) = (1 - e^{-k(T-t)})/k$ , as described in Section 3.2 of Brigo and Mercurio (2006). Therefore, the drift coefficient of the no-arbitrage bond price under  $F^T$  is  $Y_t + (1 - e^{-k(T-t)})^2 \sigma^2/k^2$ . When the horizon  $T$  becomes infinitely large, this term converges a.s. to  $Y_t + \sigma^2/k^2$ . On the contrary, the drift parameter  $r_s^T$  of the rate-adjusted price converges a.s. to the long-term yield  $r^\infty = \theta - \sigma^2/2k^2$ .

The left panel of Figure 3 shows prices  $\pi_t(1_T)$  and  $\rho_t^T(0, 1_T)$  of pure discount  $T$ -bonds when the horizon is fixed.<sup>18</sup> The rate-adjusted price overlaps the no-arbitrage price both in the short-term and in the proximity of the redemption date, consistently with the discussion in Subsections 3.1 and 3.4. Moreover, due to its simple expression, the rate-adjusted price turns out to be smoother than the risk-neutral price. In Appendix E we repeat the exercise with Cox, Ingersoll, and Ross (1985) model.

### 6.1.2 Call options on zero-coupon bonds in Vasicek (1977)

So far we discussed no-arbitrage prices of zero-coupon bonds under Vasicek assumptions. However, also derivative pricing has been extensively studied in this setting by using forward measures. A germane example is provided by Jamshidian (1989), which extends Black and Scholes (1973) approach to options on pure discount bonds.

Thus, consider a European call option with expiration  $\tau$  and strike price  $c$  over a zero-coupon bond with maturity  $T > \tau$ . When short-term rates follow Vasicek dynamics, the no-arbitrage price of the option at any time  $t$  between 0 and  $\tau$  is  $\pi_t(h_\tau) = \pi_t(1_T) \mathcal{N}(q) - c\pi_t(1_\tau) \mathcal{N}(q - \hat{\sigma})$ , where  $\mathcal{N}$  denotes the cumulative distribution function of a standard Gaussian,

$$q = \frac{\log(\pi_t(1_T)) - \log(c\pi_t(1_\tau)) + \hat{\sigma}^2/2}{\hat{\sigma}},$$

<sup>18</sup>For the numerical exercise with Vasicek model, we fix the parameters  $k = 0.5$ ,  $\theta = 0.04$  and  $\sigma = 0.01$  on a monthly time grid. We also set  $s = 0$ ,  $\xi = 0.2$  and  $Y_0 = 0.02$ . Figure 5 in Appendix D displays the term structure of no-arbitrage prices and yields of a pure discount bond.

$$\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2(k-\sigma\xi)(\tau-t)}}{2(k-\sigma\xi)} \frac{(1 - e^{-(k-\sigma\xi)(T-\tau)})^2}{(k-\sigma\xi)^2}.$$

See the derivation in Jamshidian (1989). As illustrated in Subsection 3.1, the rate-adjusted price of the option can be easily obtained from the no-arbitrage price through eq. (12).

We plot both price processes in the right panel of Figure 3, where we set  $T = 72$  months,  $\tau = 36$  months and  $c = 0.89$ . The difference between the two quantities may be appreciated only at some intermediate months.

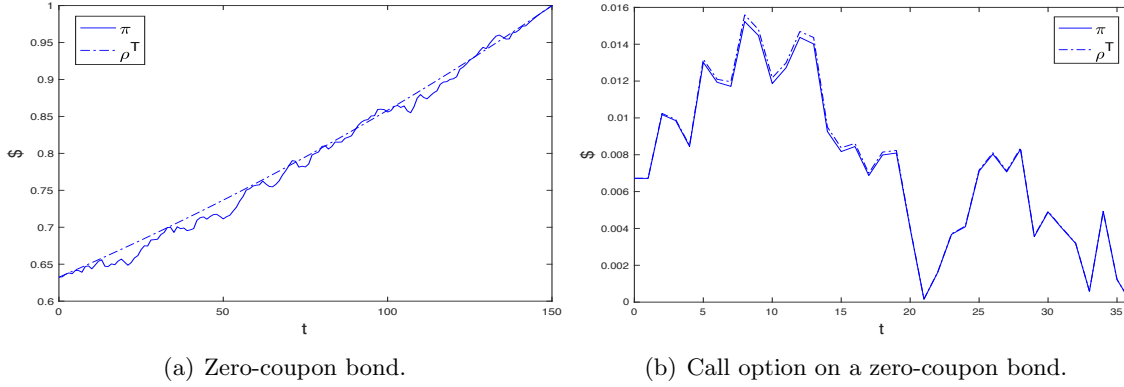


Figure 3: Left panel: values of  $\rho_t^T(0, 1_T)$  and of a realization of  $\pi_t(1_T)$  in Vasicek. Right panel: realizations of  $\rho_t^T(0, h_\tau)$  and  $\pi_t(h_\tau)$  for a European call option on a pure discount bond as described in the text. No-arbitrage prices are represented by solid lines and rate-adjusted prices by dashed lines.

### 6.1.3 Long-term relations for any payoff

We now consider a generic attainable payoff  $h_T \in L_s^1(\mathcal{F}_T, F^T)$  in a market with exponential affine interest rates. The ratio between the rate-adjusted price and the no-arbitrage price depends only on the instantaneous rates at instants  $s$  and  $t$ :

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} = e^{\{A(s,T) - B(s,T)Y_s\} \frac{T-t}{T-s} - A(t,T) + B(t,T)Y_t}.$$

In addition, the long-run relation between  $\rho^T$  and  $\pi$  of Proposition 4 can be determined explicitly under Vasicek assumptions:

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{a.s.} e^{-\frac{1}{k}(Y_s - Y_t)}, \quad T \rightarrow +\infty.$$

This limit is determined by the speed parameter  $k$ . If  $k$  is high, the two prices are almost indistinguishable for large maturities. More is actually true: for any (finite) horizon  $T$ , the ratio



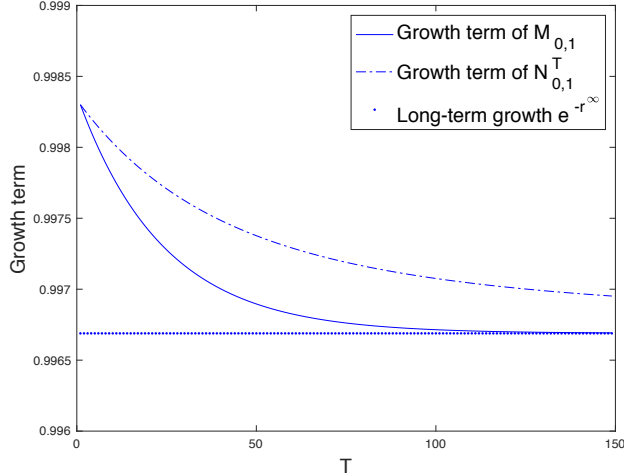


Figure 4: Term structure of growth terms of the pricing kernel  $M_{0,1}$  and the rated-adjusted pricing kernel  $N_{0,1}^T$  in Vasicek model. The solid line represents the growth term of  $M_{0,1}$  for increasing horizons  $T$ , while the dashed line regards the growth term of  $N_{0,1}^T$ . The horizontal line is the long-term growth  $e^{-r^\infty}$ .

between  $\rho_t^T$  and  $\pi_t$  converges to 1 a.s. when  $k$  grows to infinity. Indeed, fast mean reversion and reduced long-run variance make Vasicek instantaneous rates behave as if they were constant and  $\rho_t^T$  collapses to  $\pi_t$ .<sup>19</sup>

#### 6.1.4 Pricing kernel growth

We finally focus on the pricing kernel between  $s$  and  $t$  in case bond yields are affine. The growth term of  $M_{s,t}$  is an exponential function of  $Y_s$ , namely

$$\frac{e^{r_s^T s} \pi_s(1_T)}{\pi_s(1_{T-t})} = e^{A(s,T)\left(\frac{T-2s}{T-s}\right) - A(s,T-t) - \{B(s,T)\left(\frac{T-2s}{T-s}\right) - B(s,T-t)\} Y_s}.$$

When the terminal date  $T$  goes to infinity, this term converges a.s. to  $e^{-r^\infty(t-s)}$ , as expected. In Figure 4 we display this convergence under our Vasicek specifications by setting  $s = 0$  and  $t = 1$ . In addition, we graphically compare the growth term of  $M_{0,1}$  with the one of the rate-adjusted pricing kernel  $N_{0,1}^T$ . The two terms share the same long-run convergence.

## 6.2 Pricing with stocks and bonds

The situation is more involved when the market is generated by a set of risky assets with prices  $X_t^1, \dots, X_t^N$ , beyond pure discount bonds. Many sources of randomness are present, other than

<sup>19</sup>To illustrate this point, in Figure 6 in Appendix D we plot the two prices for increasing values of  $k$ , while keeping  $\theta = 0.04$ ,  $\sigma = 0.01$ ,  $\xi = 0.2$  and  $Y_0 = 0.02$  fixed.

short-term rates. Although all the drift coefficients of no-arbitrage prices of such securities coincide with  $Y_t$  under  $Q$ , price dynamics become awkward when the forward measure is employed. The drifts of these prices under  $F^T$  are additionally affected by the correlation between the instantaneous rate and the idiosyncratic random component of the asset under consideration. Rabinovitch (1989) – that incorporates the previous Merton (1973) – discusses this point at length. On the contrary, the drift of the corresponding rate-adjusted prices is always equal to the yield  $r_s^T$  of zero-coupon bonds, for any traded security. To further elucidate the issue, we borrow the dynamics of rates and stock prices from Appendix B of Brigo and Mercurio (2006).

We assume that short-term rates move as in Vasicek model, with dynamics described by eq. (27) in the time interval  $[s, T]$ . Then, we consider a stock price  $X_t$  that follows a geometric Brownian motion with volatility  $\eta > 0$ , correlated with interest rates shocks. The instantaneous correlation parameter between the two underlying Wiener processes is  $\phi$ . We can make the two sources of randomness orthogonal and consider, without loss of generality, the dynamics

$$\begin{cases} dX_t = X_t Y_t dt + \eta X_t \left( \phi dW_t^Q + \sqrt{1 - \phi^2} dZ_t^Q \right) \\ dY_t = k(\theta - Y_t) dt + \sigma dW_t^Q, \end{cases}$$

where  $W_t^Q$  and  $Z_t^Q$  are independent Wiener processes. Under the forward measure, we get

$$\begin{cases} dX_t = X_t \left( Y_t - \frac{\phi \sigma \eta}{k} (1 - e^{-k(T-t)}) \right) dt + \eta X_t \left( \phi dW_t^{F^T} + \sqrt{1 - \phi^2} dZ_t^{F^T} \right) \\ dY_t = \left( k(\theta - Y_t) - \frac{\sigma^2}{k} (1 - e^{-k(T-t)}) \right) dt + \sigma dW_t^{F^T}. \end{cases}$$

From the differential of  $X_t$  it is apparent that the correlation parameter  $\phi$  impacts on the drift of  $X_t$  under the measure  $F^T$ .

In the following subsections we derive the Feynman-Kač partial differential equations satisfied by no-arbitrage and rate-adjusted prices in this market and we determine the dynamics of the related pricing kernels.

### 6.2.1 Feynman-Kač partial differential equations

Consider a contingent claim  $h_T \in L_s^1(\mathcal{F}_T, F^T)$  which is a continuous positive function of  $X_T$  and  $Y_T$ . We now determine the Feynman-Kač partial differential equations necessarily satisfied by the two prices.<sup>20</sup>

The discounted price  $e^{-\int_s^t Y_\tau d\tau} \pi_t(h_T)$  is a  $Q$ -martingale. Therefore, by setting its drift equal to

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<sup>20</sup>We assume that  $\pi_t(h_T)$  and  $\rho_t^T(s, h_T)$  are continuously differentiable with respect to  $t$  and twice continuously differentiable with respect to the variables  $X_t = x$  and  $Y_t = y$ . For a comprehensive treatment of the topic, refer to Karatzas and Shreve (1991), Section 4.4.

zero under  $Q$ , we get the Feynman-Kač partial differential equation for  $\pi$ :

$$\frac{\partial \pi}{\partial t} + xy \frac{\partial \pi}{\partial x} + k(\theta - y) \frac{\partial \pi}{\partial y} + \frac{\eta^2 x^2}{2} \frac{\partial^2 \pi}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial^2 \pi}{\partial y^2} + \phi \sigma \eta x \frac{\partial^2 \pi}{\partial x \partial y} = y\pi \quad (28)$$

with terminal condition  $\pi_T(h_T) = h_T$ . A detailed derivation is in Appendix A. Note the role of the correlation parameter  $\phi$  in weighing the cross derivative of  $\pi$ . In case  $\phi = 0$  and interest rates are constant, the usual Black-Scholes partial differential equation arises.

Regarding the rate-adjusted price, since the forward price  $e^{r_s^T(T-t)} \rho_t^T(s, h_T)$  is an  $F^T$ -martingale, we set its drift equal to zero under  $F^T$ . Therefore, the Feynman-Kač partial differential equation for  $\rho^T$  is

$$\begin{aligned} \frac{\partial \rho^T}{\partial t} + x \left( y - \frac{\phi \sigma \eta}{k} \left( 1 - e^{-k(T-t)} \right) \right) \frac{\partial \rho^T}{\partial x} + \frac{\eta^2 x^2}{2} \frac{\partial^2 \rho^T}{\partial x^2} + \frac{\sigma^2}{2} \frac{\partial^2 \rho^T}{\partial y^2} \\ + \left( k(\theta - y) - \frac{\sigma^2}{k} \left( 1 - e^{-k(T-t)} \right) \right) \frac{\partial \rho^T}{\partial y} + \phi \sigma \eta x \frac{\partial^2 \rho^T}{\partial x \partial y} = r_s^T \rho^T \end{aligned} \quad (29)$$

with  $\rho_T^T = h_T$ . See the derivation in Appendix A.

Observe the different right-hand sides in eq. (28) and (29). They contain the instantaneous rate  $y$  for  $\pi$  and the yield to maturity  $r_s^T$  for  $\rho^T$ . The coefficients of spatial first-order derivatives are also different but the dissimilarity reduces when the speed  $k$  increases. In addition, the coefficients of  $\partial \pi / \partial x$  and  $\partial \rho^T / \partial x$  coincide when the correlation is absent.

Hence, the disparity between  $\pi$  and  $\rho^T$  can also be seized through the solution of different partial differential equations that share the same parabolic structure.

## 6.2.2 Pricing kernel dynamics

We now explicitly establish the evolution of  $M_{s,t}$  and  $N_{s,t}^T$  in our market. Beyond the money market account, the prices of the assets that generate the market satisfy

$$\begin{cases} dX_t = X_t Y_t dt + \eta X_t \left( \phi dW_t^Q + \sqrt{1 - \phi^2} dZ_t^Q \right) \\ d\pi_t(1_T) = \pi_t(1_T) Y_t dt - \pi_t(1_T) B(t, T) \sigma dW_t^Q, \end{cases}$$

where the dynamics of  $\pi_t(1_T)$  are derived as in Subsection 6.1.1. At the same time, under the physical measure,

$$\begin{cases} dX_t = X_t \mu_t^X dt + \eta X_t \left( \phi dW_t^P + \sqrt{1 - \phi^2} dZ_t^P \right) \\ d\pi_t(1_T) = \pi_t(1_T) \mu_t^P dt - \pi_t(1_T) B(t, T) \sigma dW_t^P, \end{cases}$$

where  $\mu_t^X$  and  $\mu_t^P$  are adapted processes. They are related to the drifts under  $Q$  via the bivariate process of market price of risk  $[\nu_t^W, \nu_t^Z]'$  such that

$$\left[ dW_t^Q, dZ_t^Q \right]' = \left[ \nu_t^W, \nu_t^Z \right]' dt + \left[ dW_t^P, dZ_t^P \right]'.$$

By assuming that  $\mu_t^P = (1 - \xi B(t, T)\sigma)Y_t$  for some  $\xi > 0$ , we obtain

$$\nu_t^W = \xi Y_t, \quad \nu_t^Z = \frac{\mu_t^X - Y_t - \eta\phi\nu_t^W}{\eta\sqrt{1 - \phi^2}},$$

where  $\nu_t^W$  is in line with the usual approach to Vasicek short-term rates.

Now consider the pricing kernel  $M_{s,t}$ . Since the processes defined by  $M_{s,t}H_t$ , where  $H_t$  is each of  $X_t$ ,  $\pi_t(1_T)$  and  $B_t$ , are (conditional)  $P$ -martingales in  $[s, T]$ , their drifts are null and the dynamics of  $M_{s,t}$  turns out to be

$$dM_{s,t} = -Y_t M_{s,t} dt - \nu_t^W M_{s,t} dW_t^P - \nu_t^Z M_{s,t} dZ_t^P.$$

The differential of the rate-adjusted pricing kernel  $N_{s,t}^T$  can be inferred from the multiplicative relation  $N_{s,t}^T = \pi_t(1_T)e^{r_s^T(T-t)}M_{s,t}$  by applying Itô's product rule. As a result, we have

$$dN_{s,t}^T = -r_s^T N_{s,t}^T dt - (\nu_t^W + B(t, T)\sigma)N_{s,t}^T dW_t^P - \nu_t^Z N_{s,t}^T dZ_t^P.$$

As expected, the dynamics of  $N_{s,t}^T$  coincide with the ones of  $M_{s,t}$  when interest rates are constant. Indeed,  $\sigma$  is null and the yield  $r_s^T$  coincides with the short-term rate. In general, the drift of  $N_s^T$  is driven by  $r_s^T$  in agreement with Theorem 9, while the one of  $M_{s,t}$  exploits the stochastic rate  $Y_t$ .

## 7 Conclusions

This paper generalizes to stochastic-rate markets a key property of risk-neutral pricing. Indeed, if interest rates are constant (and deterministic) over time, the instantaneous rate is both the principal eigenvalue in the return-rate relation and the stochastic discount factor growth rate. If rates are stochastic, this feature is satisfied by rate-adjusted prices that, in fact, are indistinguishable from no-arbitrage prices when rates are constant. In particular, zero-coupon bond yields replace instantaneous rates and the forward measure is employed instead of the risk-neutral one. Importantly, bond yields are able to capture the growth rate of rate-adjusted pricing kernels. This rate coincides with the one of the effective pricing kernel when the horizon under consideration is infinite. Moreover, rate-adjusted pricing kernels feature a trivial transient component in their

Hansen-Scheinkman decomposition. These two facts allow us to consider rate-adjusted prices as the prices of long-term interest rate risk.

The multiplicative decomposition of the no-arbitrage price of a security into a rate-adjusted price and a rate adjustment permits to disentangle the long-term from the short-term exposure to interest rate risk. The rate-adjusted price is associated with persistent shocks in the term structure of rates, while the adjustment captures temporary variations in the yield curve. Therefore, our theory is fruitful for the risk management of financial contracts that feature long maturities as well as interest rate risk exposure. Section 5 provides examples about derivative pricing, insurances, mortgages and pension plans. In addition, our framework may shed some light on the maturity mismatch between deposits and loans in the banking system (Hoffmann, Langfield, Pierobon, and Vuillemeys, 2019), as well as the discounting methodology of life insurance and pension companies' assets and liabilities that are subject to the European Solvency II regulation (Jørgensen, 2018).

Considering further specific dynamics of interest rates may constitute an additional avenue for future research. In a broader context, it could also be desirable to characterize the evolution of short rates through exogenous factors that determine the information structure, in order to assess the relation between these factors and the interest rate risk exposure over time.

Finally, from a theoretical perspective, a last challenge is to study the implications of random eigenvalues in the Perron-Frobenius theory that underlies the pricing kernel decomposition. Indeed, random dominating eigenvalues may be an indicator of non-deterministic steady states for economic dynamics.

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# Appendix

## Appendix to *Long-term risk with stochastic interest rates*, F. Severino.

The Appendix regards additional theoretical issues (Section A), the proofs of Sections 3 and 4 (Sections B and C), additional figures about Vasicek (1977) dynamics (Section D), Cox, Ingersoll, and Ross (1985) dynamics (Section E) and the application to optimal consumption (Section F).

## A Additional theoretical issues

### Technical assumptions

In the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , the filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  satisfies the usual conditions and is left-continuous at  $T$ . Specifically, we mean that  $\mathbb{F}$  is complete and right-continuous, i.e.  $\mathcal{F}_t = \mathcal{F}_{t+}$  for all  $t \in [0, T)$ , and  $\mathcal{F}_T = \mathcal{F}_{T-}$ .

In the whole paper, we identify random variables that coincide almost surely and we identify stochastic processes up to modifications. Moreover, we consider processes  $u : [0, T] \times \Omega \rightarrow \mathbb{R}$  that are adapted on the given filtered probability space. This requirement is equivalent to progressively measurability up to modifications, as proved, for instance, by Proposition 1.12 in Karatzas and Shreve (1991).

### Forward measures

From the seminal work of Margrabe (1978), the use of different numéraires has become quite a common practice in asset pricing, especially in option valuation. Such changes of unit of measurements have proved to be convenient especially when dealing with stochastic interest rates, as Geman (1989) and Jamshidian (1989) discuss. A comprehensive treatment is given by Geman, El Karoui, and Rochet (1995).<sup>21</sup>

Regarding the  $T$ -forward measure, the Radon-Nikodym derivative of  $F^T$  with respect to the risk-neutral measure  $Q$  is

$$F_T^T = \frac{e^{-\int_0^T Y_\tau d\tau}}{\mathbb{E} \left[ L_T e^{-\int_0^T Y_\tau d\tau} \right]} = e^{r_0^T T - \int_0^T Y_\tau d\tau},$$

$$F_t^T = \mathbb{E}_t \left[ L_{t,T} F_T^T \right] = e^{r_0^T T - r_t^T (T-t) - \int_0^t Y_\tau d\tau}$$

and we set  $F_{t,T}^T = F_T^T / F_t^T$ . The Radon-Nikodym derivative of  $F^T$  with respect to  $P$  is, then,  $G_T^T = F_T^T L_T$  and, from  $F_t^T = \mathbb{E}_t [L_{t,T} F_T^T]$ , we have

$$G_t^T = \mathbb{E}_t [G_T^T] = \mathbb{E}_t [L_T F_T^T] = L_t F_t^T.$$

As for  $t$ -bond yields, their relation with  $T$ -bond yields is expressed by the following compounding rule.

**Lemma 10** *For any  $s \leq t \leq T$ , we have  $e^{r_s^T (T-s)} = e^{r_s^t (t-s)} \mathbb{E}_s^{F^T} [e^{r_t^T (T-t)}]$ .*

**Proof of Lemma 10.** Left to the reader. ■

This simple result additionally shows that  $\mathbb{E}_s [M_{s,t}] = e^{-r_s^t (t-s)}$ .

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<sup>21</sup>Among the others, Giroto and Ortu (1996) provide a characterization of numéraire portfolios. The no-arbitrage discussion in Delbaen and Schachermayer (1995) is also remarkable.

## Weak time-derivative in $[s, T]$

Consider the conditional space  $L_s^1(\mathcal{F}_T) = \{f \in L^0(\mathcal{F}_T) : \mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)\}$ . Cerreia-Vioglio, Kupper, Maccheroni, Marinacci, and Vogelpoth (2016) show that  $L_s^1(\mathcal{F}_T)$  is an  $L^0$ -module with the multiplicative decomposition  $L_s^1(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$ . Clearly,  $L_s^1(\mathcal{F}_T)$  contains all functions  $f$  in  $L^1(\mathcal{F}_T)$ : in this case  $\mathbb{E}_s[|f|] \in L^1(\mathcal{F}_s)$ . In general, however, the conditional expectation is defined for random variables that are merely in  $L^0(\mathcal{F}_T)$ . See, for instance, Chapter II, §7 of Shiryaev (1996).

In  $L_s^1(\mathcal{F}_T)$  we use the  $L^0$ -valued metric  $d(f, g) = \mathbb{E}_s[|f - g|]$ . Accordingly, we say that a stochastic process  $u : [s, T] \rightarrow L_s^1(\mathcal{F}_T)$  is  $L_s^1$ -continuous if and only if, for all  $t \in [s, T]$ ,  $\mathbb{E}_s[|u_\tau - u_t|] \rightarrow 0$  a.s. when  $\tau \rightarrow t$ . This property is weaker than standard  $L^1$ -continuity.

Now consider the  $L^0$ -module  $\mathcal{U}_s$ . As a consequence of Tonelli's theorem, all processes in  $\mathcal{U}_s$  are such that  $\int_s^T \mathbb{E}_s[|u_\tau|]d\tau$  belongs to  $L^0(\mathcal{F}_s)$ , where the integral is computed trajectory by trajectory. In addition,  $\mathcal{U}_s$  includes all conditional (or generalized) martingales.

We now focus on weak time-differentiability in  $[s, T]$ . The space  $C_c^1((t, T), L^0(\mathcal{F}_s))$  employed in Definition 1 consists of functions  $\varphi_s : [t, T] \rightarrow L^0(\mathcal{F}_s)$  that have compact support in  $(t, T)$  and are continuously differentiable over time in the following sense: there exists a continuous function  $\psi : [t, T] \rightarrow L^0(\mathcal{F}_s)$  with compact support in  $(t, T)$  such that the pathwise integral  $\int_t^\tau \psi(z)dz$  equals  $\varphi_s(\tau)$  for all  $\tau \in [t, T]$ . For simplicity, we denote  $\psi$  by  $\varphi'_s$ .

The next proposition shows that the weak time-derivative in  $[s, T]$  is unique, up to modifications.

**Proposition 11** *Let  $u \in \mathcal{U}_s$  be weakly time-differentiable in  $[s, T]$ . Then, the weak time-derivative of  $u$  in  $[s, T]$  is unique.*

**Proof of Proposition 11.** Follow the proof of Proposition 2.2 in Marinacci and Severino (2018) by replacing the unconditional expectation with the conditional expectation with respect to  $\mathcal{F}_s$ , and the convergence in  $L^1$  with that in  $L_s^1$ . ■

We finally prove that a weakly time-differentiable process has null weak time-derivative in  $[s, T]$  if and only if it is a conditional martingale.

**Proposition 12** *A process  $u$  belongs to  $\mathcal{U}_s^1$  and has  $\mathcal{D}u = 0$  if and only if it is a conditional martingale.*

**Proof of Proposition 12.** Suppose that  $u$  is a conditional martingale. Then,  $u$  belongs to  $\mathcal{U}_s$ . Moreover, fixed  $t \in [s, T]$ , for any  $\varphi_s \in C_c^1((t, T), L^0(\mathcal{F}_s))$  and  $A_t \in \mathcal{F}_t$ ,

$$\int_t^T \mathbb{E}_s[u_\tau \mathbf{1}_{A_t}] \varphi'_s(\tau) d\tau = \int_t^T \mathbb{E}_s[u_t \mathbf{1}_{A_t}] \varphi'_s(\tau) d\tau = \mathbb{E}_s[u_t \mathbf{1}_{A_t}] \int_t^T \varphi'_s(\tau) d\tau = 0$$

because  $\varphi_s$  is in  $C_c^1((t, T), L^0(\mathcal{F}_s))$ . Hence,  $w(t) = 0$  for any  $t \in [s, T]$  satisfies the definition of weak time-derivative of  $u$  in  $[s, T]$  and so  $\mathcal{D}u = 0$ .

Conversely, assume that  $u \in \mathcal{U}_s^1$  has  $\mathcal{D}u = 0$ . First,  $u$  is adapted and any  $u_\tau \in L_s^1(\mathcal{F}_\tau)$ . Therefore,  $\mathbb{E}_s[|u_\tau|] \in L^0(\mathcal{F}_s)$  for all  $\tau \in [s, T]$ . As a consequence,  $\mathbb{E}_t[|u_\tau|] \in L^0(\mathcal{F}_t)$  for all  $s \leq t \leq \tau \leq T$ . Indeed, since  $|u_\tau|$  is non-negative,  $\mathbb{E}_t[|u_\tau|]$  is always defined as  $\mathcal{F}_t$ -measurable extended random variable. However, if there existed a set  $A_t \in \mathcal{F}_t$  such that  $\mathbb{E}_t[|u_\tau|] \mathbf{1}_{A_t}$  equals infinity, then, taken any  $B_s \in \mathcal{F}_s$  with non-empty  $A_t \cap B_s$ ,  $\mathbb{E}_s[|u_\tau| \mathbf{1}_{B_s}] \geq \mathbb{E}_s[\mathbb{E}_t[|u_\tau|] \mathbf{1}_{A_t \cap B_s}]$  which is infinite. This fact would contradict  $u_\tau \in L_s^1(\mathcal{F}_\tau)$ .

Hence, in order to prove that  $u$  is a conditional martingale, we are just left to show that  $u$  satisfies the martingale property. We begin with proving that, given  $t \in [s, T]$ ,  $\mathbb{E}_t[u_\tau]$  is not dependent on  $\tau$  for a.e.  $\tau \in [t, T]$ .

Take into consideration a continuous function  $\eta : [t, T] \rightarrow \mathbb{R}$  with compact support in  $(t, T)$  such that  $\int_t^T \eta(\tau) d\tau = 1$ . Given a continuous function  $\xi : [t, T] \rightarrow \mathbb{R}$  with compact support in  $(t, T)$ , we define the function  $k_\xi : [t, T] \rightarrow \mathbb{R}$  by  $k_\xi(\sigma) = \xi(\sigma) - \left(\int_t^T \xi(\tau) d\tau\right) \eta(\sigma)$ .

The function  $k_\xi$  is continuous with compact support in  $(t, T)$  and  $\int_t^T k_\xi(\tau) d\tau = 0$ . Thus,  $k_\xi$  has a primitive  $K_\xi$  that is continuous with compact support in  $(t, T)$ . Since  $K_\xi \in C_c^1((t, T), \mathbb{R})$ , it is included in  $C_c^1((t, T), L^0(\mathcal{F}_s))$  and so we use it as a test function in the definition of weak time-derivative of  $u$  in  $[s, T]$ . Since  $\mathcal{D}u = 0$ , for any  $A_t \in \mathcal{F}_t$  the following holds:

$$\begin{aligned} 0 &= \int_t^T \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] \left( \xi(\sigma) - \left( \int_t^T \xi(\tau) d\tau \right) \eta(\sigma) \right) d\sigma \\ &= \int_t^T \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] \xi(\sigma) d\sigma - \int_t^T \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] \left( \int_t^T \xi(\tau) d\tau \right) \eta(\sigma) d\sigma \\ &= \int_t^T \mathbb{E}_s [u_\tau \mathbf{1}_{A_t}] \xi(\tau) d\tau - \int_t^T \left( \int_t^T \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] \eta(\sigma) d\sigma \right) \xi(\tau) d\tau \\ &= \int_t^T \left( \mathbb{E}_s [u_\tau \mathbf{1}_{A_t}] - \int_t^T \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] \eta(\sigma) d\sigma \right) \xi(\tau) d\tau. \end{aligned}$$

By Lemma A.1 in the Appendix of Marinacci and Severino (2018), for a.e.  $\tau \in [t, T]$

$$\mathbb{E}_s [u_\tau \mathbf{1}_{A_t}] = \int_t^T \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] \eta(\sigma) d\sigma.$$

Then, since  $\int_t^T \eta(\sigma) d\sigma = 1$ , we get

$$\int_t^T (\mathbb{E}_s [u_\tau \mathbf{1}_{A_t}] - \mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}]) \eta(\sigma) d\sigma = 0.$$

As the last equality is satisfied by any continuous function  $\eta$  with compact support in  $(t, T)$ , it follows that, for a.e.  $\sigma \in [t, T]$ ,  $\mathbb{E}_s [u_\sigma \mathbf{1}_{A_t}] = \mathbb{E}_s [u_\tau \mathbf{1}_{A_t}]$  and so  $\mathbb{E}_t [u_\sigma] = \mathbb{E}_t [u_\tau]$ . Consequently,  $\mathbb{E}_t [u_\tau]$  is not dependent on  $\tau$  for a.e.  $\tau \in [t, T]$  and so  $\mathbb{E}_t [u_\tau] = f_t$  for some  $f_t \in L_s^1(\mathcal{F}_t)$ .

$u$  is  $L_s^1$ -right-continuous and so  $\mathbb{E}_t [u_\tau]$  goes to  $u_t$  in  $L_s^1$  when  $\tau \rightarrow t^+$ . Since for a.e.  $\tau \in [t, T]$ ,  $\mathbb{E}_t [u_\tau]$  coincides a.s. with  $f_t$ , which does not depend on  $\tau$ , the uniqueness of the  $L_s^1$ -limit ensures that  $f_t = u_t$ . Therefore, for any  $t \in [0, T]$  and for a.e.  $\tau \in [t, T]$ ,  $\mathbb{E}_t [u_\tau] = u_t$ .

The last property is actually satisfied by any  $\tau \in [t, T]$ . Indeed, fix any  $\tau$  and consider a sequence  $\{\tau_i\}_{i \in \mathbb{N}} \subset [t, T]$  such that  $\tau_i \rightarrow \tau^+$  and  $\mathbb{E}_t [u_{\tau_i}] = u_t$ . Since  $u$  is  $L_s^1$ -right-continuous, the  $L_s^1$ -limit of  $\mathbb{E}_t [u_{\tau_i}]$  is  $\mathbb{E}_t [u_\tau]$ . Nevertheless,  $\mathbb{E}_t [u_{\tau_i}] = u_t$  for all  $i$  and so, by uniqueness of the  $L_s^1$ -limit,  $\mathbb{E}_t [u_\tau] = u_t$ . ■

## Derivation of Feynman-Kac partial differential equations

We take into consideration a payoff  $h_T$  as in Subsection 6.2.1 and we apply Itô's product rule to the discounted no-arbitrage price  $e^{-\int_s^t Y_\tau d\tau} \pi_t(h_T)$ . By omitting negligible higher order terms, we get

$$d \left( e^{-\int_s^t Y_\tau d\tau} \pi_t(h_T) \right) = e^{-\int_s^t Y_\tau d\tau} (d\pi_t(h_T) - Y_t \pi_t(h_T) dt).$$

Moreover,

$$\begin{aligned} d\pi_t(h_T) &= \frac{\partial\pi}{\partial t}dt + \frac{\partial\pi}{\partial x}X_t \left( Y_t dt + \eta\phi dW_t^Q + \eta\sqrt{1-\phi^2}dZ_t^Q \right) \\ &\quad + \frac{\partial\pi}{\partial y} \left( k(\theta - Y_t) dt + \sigma dW_t^Q \right) \\ &\quad + \frac{\partial^2\pi}{\partial x^2} \frac{\eta^2 X_t^2}{2} dt + \frac{\partial^2\pi}{\partial y^2} \frac{\sigma^2}{2} dt + \frac{\partial^2\pi}{\partial x\partial y} \phi\sigma\eta X_t dt. \end{aligned}$$

Since  $e^{-\int_s^t Y_\tau d\tau} \pi_t(h_T)$  defines a  $Q$ -martingale, the drift of its differential is null. As a result, by setting  $X_t = x$  and  $Y_t = y$ , we obtain the Feynman-Kač partial differential equation for the no-arbitrage price, i.e. eq. (28).

Regarding the rate-adjusted price of  $h_T$ , we have

$$d\left(e^{r_s^T(T-t)}\rho_t^T(s, h_T)\right) = e^{r_s^T(T-t)} \left(d\rho_t^T(s, h_T) - r_s^T\rho_t^T dt\right)$$

and

$$\begin{aligned} d\rho_t^T(s, h_T) &= \frac{\partial\rho^T}{\partial t}dt + \frac{\partial\rho^T}{\partial x}X_t \left( \left( Y_t - \frac{\phi\sigma\eta}{k} \left( 1 - e^{-k(T-t)} \right) \right) dt \right. \\ &\quad \left. + \eta\phi dW_t^{F^T} + \eta\sqrt{1-\phi^2}dZ_t^{F^T} \right) \\ &\quad + \frac{\partial\rho^T}{\partial y} \left( \left( k(\theta - Y_t) - \frac{\sigma^2}{k} \left( 1 - e^{-k(T-t)} \right) \right) dt + \sigma dW_t^{F^T} \right) \\ &\quad + \frac{\partial^2\rho^T}{\partial x^2} \frac{\eta^2 X_t^2}{2} dt + \frac{\partial^2\rho^T}{\partial y^2} \frac{\sigma^2}{2} dt + \frac{\partial^2\rho^T}{\partial x\partial y} \phi\sigma\eta X_t dt. \end{aligned}$$

By setting to zero the drift of the differential of the forward price process, i.e.  $e^{r_s^T(T-t)}\rho_t^T(s, h_T)$ , which is an  $F^T$ -martingale, we deduce the Feynman-Kač partial differential equation for  $\rho^T$ , namely eq. (29).

## B Proofs of Section 3

### Proof of Theorem 2

(*Existence*) In order to show that  $\rho^T \in \mathcal{U}_s^1$ , we prove that  $e^{r_s^T T} \rho^T$  belongs to  $\mathcal{U}_s$  and is weakly time-differentiable in  $[s, T]$ .

First, for all  $\tau \in [s, T]$ ,  $e^{r_s^T T} \rho_\tau^T \in L_s^1(\mathcal{F}_\tau)$ . Indeed,  $|e^{r_s^T T} \rho_\tau^T|$  is non-negative and so its conditional expectation at time  $s$  is an extended real random variable. However,  $\mathbb{E}_s^{F^T} [|e^{r_s^T T} \rho_\tau^T|] \leq e^{r_s^T \tau} \mathbb{E}_s^{F^T} [|h_T|]$ , which is in  $L^0(\mathcal{F}_s)$ . Thus,  $\mathbb{E}_s^{F^T} [|e^{r_s^T T} \rho_\tau^T|]$  is in  $L^0(\mathcal{F}_s)$ .

Regarding  $L_s^1$ -continuity, we check that, for any  $t \in [s, T)$ ,  $\mathbb{E}_s^F [|e^{r_s^T T} \rho_\tau^T - e^{r_s^T T} \rho_t^T|]$  tends to zero when

$\tau \rightarrow t^+$ . We have

$$\begin{aligned}
& \mathbb{E}_s^{F^T} \left[ \left| e^{r_s^T T} \rho_\tau^T - e^{r_s^T t} \rho_t^T \right| \right] = e^{r_s^T t} \mathbb{E}_s^{F^T} \left[ \left| e^{r_s^T (\tau-t)} \mathbb{E}_\tau^{F^T} [h_T] - \mathbb{E}_t^{F^T} [h_T] \right| \right] \\
& \leq e^{r_s^T t} \left( \mathbb{E}_s^{F^T} \left[ \left| e^{r_s^T (\tau-t)} \mathbb{E}_\tau^{F^T} [h_T] - \mathbb{E}_\tau^{F^T} [h_T] \right| \right] \right. \\
& \quad \left. + \mathbb{E}_s^{F^T} \left[ \left| \mathbb{E}_\tau^{F^T} [h_T] - \mathbb{E}_t^{F^T} [h_T] \right| \right] \right) \\
& \leq e^{r_s^T t} \left( \left| e^{r_s^T (\tau-t)} - 1 \right| \mathbb{E}_s^{F^T} [|h_T|] + \mathbb{E}_s^{F^T} \left[ \left| \mathbb{E}_\tau^{F^T} [h_T] - \mathbb{E}_t^{F^T} [h_T] \right| \right] \right).
\end{aligned}$$

In the last expression, both addends go to zero a.s. when  $\tau$  approaches  $t^+$ . In particular, the convergence of the first one follows from the fact that almost every realization of  $r_s^T$  is a real number (fixed for the convergence). As to the second term, its convergence is ensured by Lévy's downward theorem that guarantees that  $\mathbb{E}_\tau^{F^T} [h_T]$  goes in  $L_s^1$  to  $\mathbb{E}_{t^+}^{F^T} [h_T] = \mathbb{E}_t^{F^T} [h_T]$  when  $\tau \rightarrow t^+$ . Similarly, when  $\tau \rightarrow T^-$ , the convergence is due to Lévy's upward theorem. Therefore  $e^{r_s^T T} \rho^T$  belongs to  $\mathcal{U}_s$ . To be precise, as described in Chapter 14 of Williams (1991), Lévy's theorems require that  $h_T \in L^1(\mathcal{F}_T)$  and ensure the previous convergences in  $L^1$  of conditional expectations. However, these results still hold when  $h_T \in L_s^1(\mathcal{F}_T)$  by using the convergence in  $L_s^1$ . Indeed, as shown in Cerreia-Vioglio, Kupper, Maccheroni, Marinacci, and Vogelpoth (2016),  $L_s^1(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$  and so  $h_T = a_s k_T$  for some  $a_s \in L^0(\mathcal{F}_s)$  and  $k_T \in L^1(\mathcal{F}_T)$ . Then, Lévy's theorems apply to  $k_T$  with the convergence in  $L^1$ . The latter implies the convergence in  $L_s^1$ , which holds also after multiplying  $k_T$  by  $a_s$  and retrieving  $h_T$ .

Now we look for the weak time-derivative in  $[s, T]$  of  $e^{r_s^T T} \rho^T$ . We consider any  $A_t \in \mathcal{F}_t$  and  $\varphi_s \in C_c^1((t, T), L^0(\mathcal{F}_s))$ . Since indicator functions  $\mathbf{1}_{A_t}$  are  $\mathcal{F}_\tau$ -measurable for all  $\tau \in [t, T]$ ,

$$\begin{aligned}
& - \int_t^T \mathbb{E}_s^{F^T} \left[ e^{r_s^T T} \rho_\tau^T \mathbf{1}_{A_t} \right] \varphi'_s(\tau) d\tau = - \int_t^T e^{r_s^T \tau} \mathbb{E}_s^{F^T} [h_T \mathbf{1}_{A_t}] \varphi'(\tau) d\tau \\
& = - \mathbb{E}_s^{F^T} [h_T \mathbf{1}_{A_t}] \int_t^T e^{r_s^T \tau} \varphi'_s(\tau) d\tau = \mathbb{E}_s^{F^T} [h_T \mathbf{1}_{A_t}] \int_t^T r_s^T e^{r_s^T \tau} \varphi_s(\tau) d\tau \\
& = \int_t^T r_s^T \mathbb{E}_s^{F^T} \left[ e^{r_s^T \tau} h_T \mathbf{1}_{A_t} \right] \varphi_s(\tau) d\tau = \int_t^T \mathbb{E}_s^{F^T} \left[ r_s^T e^{r_s^T T} \rho_\tau^T \mathbf{1}_{A_t} \right] \varphi_s(\tau) d\tau.
\end{aligned}$$

The integral of the function  $\sigma \mapsto e^{r_s^T \sigma} \varphi'_s(\sigma)$  is computed pathwise in  $L^0(\mathcal{F}_s)$ , exploiting the compact support of  $\varphi_s$ .

Therefore, the candidate weak time-derivative in  $[s, T]$  of  $e^{r_s^T T} \rho^T$  is  $r_s^T e^{r_s^T T} \rho^T$ . Since  $r_s^T e^{r_s^T T} \rho^T$  belongs to  $\mathcal{U}_s$ , we can claim that  $\mathcal{D}\rho^T = r_s^T \rho^T$ . Of course,  $\rho_T^T = h_T$  and so  $\rho^T \in \mathcal{U}_s^1$  solves problem (10).

(Uniqueness) Let  $f^{(1)}, f^{(2)} \in \mathcal{U}_s^1$  be two solutions of problem (10) and define  $z = f^{(1)} - f^{(2)} \in \mathcal{U}_s^1$ . We have that  $\mathcal{D}z = r_s^T z$  and  $z_T = 0$ . We now compute the weak time-derivative of  $e^{-r_s^T t} z_t$  in  $[s, T]$ . Fix  $t \in [s, T]$ . For any  $\varphi_s \in C_c^1((t, T), L^0(\mathcal{F}_s))$ , consider the function

$$\theta \mapsto e^{-r_s^T \theta} r_s^T \varphi_s(\theta) - e^{-r_s^T \theta} \varphi'_s(\theta)$$

that takes values in  $L^0(\mathcal{F}_s)$ . By integrating pathwise, it follows that

$$\int_\tau^T \left( e^{-r_s^T \theta} r_s^T \varphi_s(\theta) - e^{-r_s^T \theta} \varphi'_s(\theta) \right) d\theta = e^{-r_s^T \tau} \varphi_s(\tau).$$

Hence,  $e^{-r_s^T \tau} \varphi_s(\tau)$  belongs to  $C_c^1((t, T), L^0(\mathcal{F}_s))$  and so we can use it as test function in the definition of weak time-derivative of  $z$  in  $[s, T]$ :

$$\begin{aligned} & \int_t^T \mathbb{E}_s^{F^T} \left[ \mathcal{D}z_\tau \mathbf{1}_{A_t} e^{-r_s^T \tau} \right] \varphi_s(\tau) d\tau \\ &= - \int_t^T \mathbb{E}_s^{F^T} [z_\tau \mathbf{1}_{A_t}] \left( e^{-r_s^T \tau} \varphi_s'(\tau) - e^{-r_s^T \tau} r_s^T \varphi_s(\tau) \right) d\tau \\ &= - \int_t^T \mathbb{E}_s^{F^T} [z_\tau \mathbf{1}_{A_t} e^{-r_s^T \tau}] \varphi_s'(\tau) d\tau + \int_t^T \mathbb{E}_s^{F^T} [z_\tau \mathbf{1}_{A_t} e^{-r_s^T \tau} r_s^T] \varphi_s(\tau) d\tau. \end{aligned}$$

Consequently, the weak time-derivative of  $e^{-r_s^T t} z_t$  in  $[s, T]$  is  $e^{-r_s^T t} (\mathcal{D}z_t - r_s^T z_t)$ .

However this process is null. Therefore,  $e^{-r_s^T t} z_t$  has null weak time-derivative in  $[s, T]$ . Hence, by Proposition 12,  $e^{-r_s^T t} z_t$  is a conditional  $F^T$ -martingale and so, for any  $t \in [s, T]$  and  $\tau \in [t, T]$ , we have  $\mathbb{E}_t^{F^T} [z_\tau] = e^{r_s^T (\tau-t)} z_t$ . When  $\tau$  goes to  $T^-$ , we get that  $\mathbb{E}_t^{F^T} [z_\tau]$  tends to  $e^{r_s^T (T-t)} z_t$  pointwise.

In addition,  $z_\tau$  converges to  $z_T = 0$  in  $L_s^1$  as  $\tau$  approaches  $T^-$  and so  $\mathbb{E}_t^{F^T} [z_\tau]$  tends to zero in  $L_s^1$ . By uniqueness of the  $L_s^1$ -limit,  $z_t = 0$  for all  $t \in [s, T]$ . This proves uniqueness of the solution of problem (10).

### Proof of Proposition 3

Fix any positive  $s$  and consider the limit in probability when  $T$  goes to infinity under Assumptions 1. By Theorem 3.2 in Qin and Linetsky (2017),  $r_s^T \xrightarrow{P} r^\infty$ . Since  $r^\infty$  is positive,  $e^{-r_s^T (T-t)} \xrightarrow{P} 0$  for all  $s > 0$  and  $t > s$ . Therefore,  $e^{-r_s^T (T-t)} - e^{-r_t^T (T-t)} \xrightarrow{P} 0$  and so  $(\rho_t^T(s, h_T) - \pi_t(h_T)) / \mathbb{E}_t^{F^T} [h_T]$  tends to zero.

As for the second convergence, suppose that  $h_T$  is strictly positive. Since  $r_s^T \xrightarrow{P} r^\infty$  for all  $s > 0$  as  $T$  goes to infinity, the difference  $r_s^T - r_t^T$  converges in probability to zero for all  $s > 0$  and  $t > s$ . In addition, for any positive  $\varepsilon$ ,

$$P \left( \frac{|\log \rho_t^T(s, h_T) - \log \pi_t(h_T)|}{T-t} > \varepsilon \right) = P(|r_s^T - r_t^T| > \varepsilon)$$

and this quantity goes to zero because  $r_s^T - r_t^T \xrightarrow{P} 0$  as  $T$  increases.

### Proof of Proposition 4

By Lemma 10 in Appendix A, for any  $t > s$

$$\mathbb{E}_s^{F^T} \left[ \frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \right] = e^{r_s^T (t-s)} \mathbb{E}_s^{F^T} \left[ \frac{\pi_s(1_T)}{\pi_t(1_T)} \right] = e^{r_s^T (t-s)} \pi_s(1_t) \xrightarrow{P} e^{(r^\infty - r_s^t)(t-s)}$$

as  $T$  goes to infinity.

As to the second convergence, consider the expression

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} = e^{r_s^T (t-s)} \frac{\pi_s(1_T)}{\pi_t(1_T)} = e^{r_s^T (t-s)} \frac{\pi_0(1_{T-s})}{\pi_0(1_{T-t})} \frac{\pi_s(1_T)}{\pi_0(1_{T-s})} \frac{\pi_0(1_{T-t})}{\pi_t(1_T)}$$

and recall Assumptions 1. Theorem 3.2 in Qin and Linetsky (2017) ensures that, as  $T$  goes to infinity,

- $e^{r_s^T (t-s)}$  converges to  $e^{r^\infty (t-s)}$  in probability;
- $\pi_0(1_{T-s}) / \pi_0(1_{T-s-(t-s)})$  converges to  $e^{-r^\infty (t-s)}$  in probability;
- $\pi_s(1_T) / \pi_0(1_{T-s})$  converges to  $b_s^\infty$  in the semimartingale topology of Émery (1979);

- $\pi_0(1_{T-t})/\pi_t(1_T)$  converges to  $1/b_t^\infty$  in the semimartingale topology.

In the semimartingale topology the product of convergent processes converges to the product of the respective limit processes. Moreover, the convergence in the semimartingale topology necessarily entails the convergence in probability for any fixed  $t$ . Therefore, by Slutsky's theorem we can conclude that

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{e^{r^\infty(t-s)} b_s^\infty}{e^{r^\infty(t-s)} b_t^\infty} = \frac{b_s^\infty}{b_t^\infty}, \quad T \rightarrow +\infty.$$

## Proof of Theorem 5

Following the proof of Theorem 2, it is easy to establish that the unique solution in  $\mathcal{U}_s^1$  is given by the process defined, at any instant  $t$ , by  $e^{-r^\infty(T-t)} \mathbb{E}_t^{F^\infty}[h_T]$ . The forward measure and the yield in  $[s, T]$  need to be replaced by  $F^\infty$  and  $r^\infty$ , respectively.

## Proof of Proposition 6

Fix  $t \in [s, \tau]$ . As we will show in Proposition 7, when  $T$  goes to infinity,  $G_{t,\tau}^T$  converges in probability to  $G_{t,\tau}^\infty$ . Therefore,  $G_{t,\tau}^T h_\tau$  goes to  $G_{t,\tau}^\infty h_\tau$  in probability. Since  $G_{t,\tau}^T h_\tau$  is also convergent in  $L^1(P)$  and this convergence implies the one in probability, by uniqueness of the limit,  $G_{t,\tau}^T h_\tau$  tends to  $G_{t,\tau}^\infty h_\tau$  in  $L^1(P)$ . Consequently,

$$\mathbb{E}_t^{F^T}[h_\tau] = \mathbb{E}_t[G_{t,\tau}^T h_\tau] \xrightarrow{L^1} \mathbb{E}_t[G_{t,\tau}^\infty h_\tau] = \mathbb{E}_t^{F^\infty}[h_\tau], \quad T \rightarrow +\infty$$

and the convergence is also in probability. In addition,  $e^{-r_s^T(\tau-t)} \xrightarrow{P} e^{-r^\infty(\tau-t)}$  and so, by the continuous mapping theorem, when  $T$  goes to infinity,

$$\rho_t^T(s, h_\tau) = e^{-r_s^T(\tau-t)} \mathbb{E}_t^{F^T}[h_\tau] \xrightarrow{P} e^{-r^\infty(\tau-t)} \mathbb{E}_t^{F^\infty}[h_\tau] = \rho_t^\infty(s, h_\tau).$$

## C Proofs of Section 4

### Proof of Proposition 7

Assumptions 1 hold and we exploit Theorem 3.2 in Qin and Linetsky (2017). Since  $T$  goes to infinity, we assume that  $T > t + s$  without loss of generality.

As for the first convergence, consider

$$\frac{e^{r_s^T s} \pi_s(1_T)}{\pi_s(1_{T-t})} = e^{r_s^T s} \frac{\pi_0(1_{T-s})}{\pi_0(1_{T-t-s})} \frac{\pi_s(1_T)}{\pi_0(1_{T-s})} \frac{\pi_0(1_{T-t-s})}{\pi_s(1_{T-t})}.$$

When  $T$  goes to infinity, we have

- $e^{r_s^T s}$  converges to  $e^{r^\infty s}$  in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-t-s})$  converges to  $e^{-r^\infty t}$  in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$  converges to  $b_s^\infty$  in the semimartingale topology;
- $\pi_0(1_{T-t-s})/\pi_s(1_{T-t})$  converges to  $1/b_s^\infty$  in the semimartingale topology.

As a result, the first convergence of the statement obtains.

Similarly,

$$\frac{e^{-r_s^T s} \pi_s(1_{T-t})}{\pi_t(1_T)} = e^{-r_s^T s} \frac{\pi_0(1_{T-t-s})}{\pi_0(1_{T-t})} \frac{\pi_s(1_{T-t})}{\pi_0(1_{T-t-s})} \frac{\pi_0(1_{T-t})}{\pi_t(1_T)}$$

and, when  $T$  goes to infinity,

- $e^{-r_s^T s}$  converges to  $e^{-r^\infty s}$  in probability;
- $\pi_0(1_{T-t-s})/\pi_0(1_{T-t})$  converges to  $e^{r^\infty s}$  in probability;
- $\pi_s(1_{T-t})/\pi_0(1_{T-t-s})$  converges to  $b_s^\infty$  in the semimartingale topology;
- $\pi_0(1_{T-t})/\pi_t(1_T)$  converges to  $1/b_t^\infty$  in the semimartingale topology.

Consequently, the second convergence of the proposition is established.

In addition, Theorem 3.2 in Qin and Linetsky (2017) ensures that  $M_{s,t} = e^{-r^\infty(t-s)}(b_s^\infty/b_t^\infty)G_{s,t}^\infty$ . At finite horizons we have

$$G_{s,t}^T = \left( \frac{e^{r_s^T s} \pi_s(1_T)}{\pi_s(1_{T-t})} \right)^{-1} \left( \frac{e^{-r_s^T s} \pi_s(1_{T-t})}{\pi_t(1_T)} \right)^{-1} M_{s,t}.$$

Since the first factor converges in probability to  $e^{r^\infty(t-s)}$  and the second one tends to  $b_t^\infty/b_s^\infty$ , we deduce that  $G_{s,t}^T$  converges in probability to  $G_{s,t}^\infty$  when  $T$  goes to infinity.

## Proof of Proposition 8

From the decomposition of  $M_{s,t}$ , we have that

$$N_{s,t}^T = e^{(r_s^T - r_t^T)(T-t)} e^{-r^\infty(t-s)} \frac{b_s^\infty}{b_t^\infty} G_{s,t}^\infty.$$

The quantity  $e^{(r_s^T - r_t^T)(T-t)}$  coincides with the ratio  $\pi_t(h_T)/\rho_t^T(s, h_T)$  when an arbitrary payoff  $h_T$  is considered. Thus, by Proposition 4,  $e^{(r_s^T - r_t^T)(T-t)}$  converges in probability to  $b_t^\infty/b_s^\infty$  as  $T$  goes to infinity, ensuring the convergence of  $N_{s,t}^T$ .

## Proof of Theorem 9

*Problem (23).* We study the properties of the process defined, at any time  $t$ , by  $G_s^T e^{r_s^T s} N_{s,t}^T$  that coincides with  $e^{-r_s^T t} G_t^T$ . Its conditional expectation at time  $s$  under the measure  $P$  belongs to  $L^0(\mathcal{F}_s)$ . Moreover,  $L_s^1$ -right-continuity at  $t$  is due to the fact that

$$\begin{aligned} \mathbb{E}_s \left[ \left| e^{-r_s^T \tau} G_\tau^T - e^{-r_s^T t} G_t^T \right| \right] &\leq e^{-r_s^T t} \left( \left| e^{-r_s^T(\tau-t)} - 1 \right| G_s^T \right. \\ &\quad \left. + \mathbb{E}_s \left[ \left| \mathbb{E}_\tau [G_\tau^T] - \mathbb{E}_t [G_t^T] \right| \right] \right) \end{aligned}$$

for all  $\tau \geq t$ . Similarly to the proof of Theorem 2, Lévy's downward theorem ensures the convergence to zero when  $\tau \rightarrow t^+$ . A parallel reasoning guarantees  $L_s^1$ -left-continuity at  $T$  and so  $e^{-r_s^T t} G_t^T$  belongs to  $\mathcal{U}_s$ .

Next, we show that the weak time-derivative in  $[s, T]$  of  $e^{-r_s^T t} G_t^T$  is  $-r_s^T e^{-r_s^T t} G_t^T$  under the physical



measure. By considering any  $A_t \in \mathcal{F}_t$  and  $\varphi_s \in C_c^1((t, T), L^0(\mathcal{F}_s))$ , we have

$$\begin{aligned} -\int_t^T \mathbb{E}_s \left[ e^{-r_s^T \tau} G_\tau^T \mathbf{1}_{A_t} \right] \varphi_s'(\tau) d\tau &= -\mathbb{E}_s \left[ G_T^T \mathbf{1}_{A_t} \right] \int_t^T e^{-r_s^T \tau} \varphi_s'(\tau) d\tau \\ &= -\mathbb{E}_s \left[ G_T^T \mathbf{1}_{A_t} \right] \int_t^T r_s^T e^{-r_s^T \tau} \varphi_s(\tau) d\tau \\ &= \int_t^T \mathbb{E}_s \left[ -r_s^T e^{-r_s^T \tau} G_\tau^T \mathbf{1}_{A_t} \right] \varphi_s(\tau) d\tau. \end{aligned}$$

Moving back to  $N_{s,t}^T$ , we established that  $N_{s,t}^T$  belongs to  $\mathcal{U}_s^1$  and  $\mathcal{D}N_{s,t}^T = -r_s^T N_{s,t}^T$ .

*Problem (24).* It is convenient to consider the process defined, at any  $t$ , by  $e^{r_s^\infty s} G_s^\infty N_{s,t}^\infty$ , that is  $e^{-r_s^\infty t} G_t^\infty$ . Since  $G_t^\infty$  is a martingale under  $P$ ,  $\mathbb{E}_s[e^{-r_s^\infty t} G_t^\infty] = e^{-r_s^\infty t} G_s^\infty$  belongs to  $L^0(\mathcal{F}_s)$ . In addition,  $\int_s^{+\infty} \mathbb{E}_s[e^{-r_s^\infty \tau} G_\tau^\infty] d\tau = G_s^\infty e^{-r_s^\infty s} / r_s^\infty$  is in  $L^0(\mathcal{F}_s)$ , too.

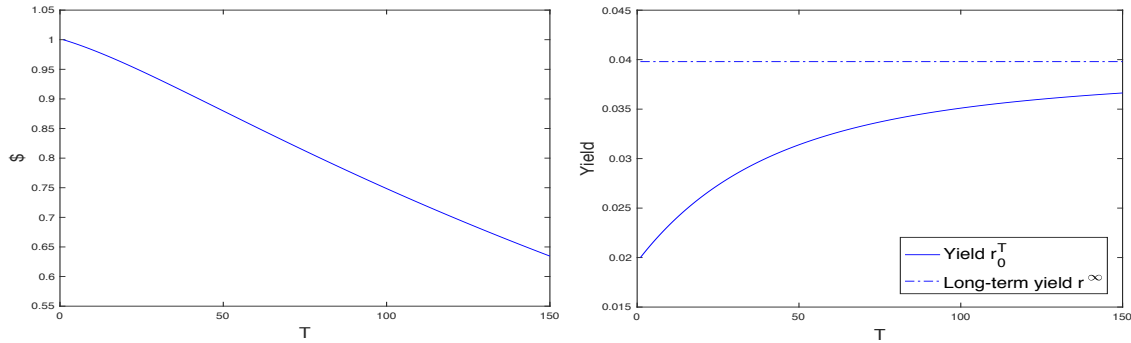
$L_s^1$ -right-continuity at any  $t$  can be shown as in the proof of problem (23) by observing that  $G_\tau^\infty = \mathbb{E}_\tau[G_T^\infty]$  and  $G_t^\infty = \mathbb{E}_t[G_T^\infty]$  for any  $T$  larger than  $\tau$ . As a result,  $e^{-r_s^\infty t} G_t^\infty$  is in  $\mathcal{U}_s$  with  $T = +\infty$ .

Regarding weak time-differentiability in  $[s, +\infty)$ , we can follow again the proof of problem (23) by integrating on intervals  $[t, +\infty)$ . Indeed, it is enough to use the relation  $G_\tau^\infty = \mathbb{E}_\tau[G_T^\infty]$ , where  $T$  is a time index larger than any instant in the (bounded) supports of  $\varphi_s$  and  $\varphi_s'$ . Thus, the weak time-derivative in  $[s, +\infty)$  of  $e^{-r_s^\infty t} G_t^\infty$  is  $-r_s^\infty e^{-r_s^\infty t} G_t^\infty$ .

Consequently,  $N_{s,t}^\infty$  turns out to be a process in  $\mathcal{U}_s^1$  with  $T = +\infty$  that satisfies  $\mathcal{D}N_{s,t}^\infty = -r_s^\infty N_{s,t}^\infty$ .

## D Additional figures about Vasicek (1977) dynamics

We plot some figures that complement the analysis of Subsections 6.1.1 and 6.1.3 about Vasicek (1977) dynamics of interest rates. The parameters employed in the simulations are mentioned in Subsection 6.1.1. Figure 5 contains the term structure of zero-coupon bond prices and the related yield curve. Figure 6 represents the no-arbitrage prices and the rate-adjusted prices of a pure discount bond when the parameter  $k$  increases.



(a) Zero-coupon bond prices term structure.

(b) Yield-to-maturity term structure.

Figure 5: The left and right panel depict no-arbitrage pure discount bond prices (at time zero) and yields to maturity with respect to increasing horizons  $T$  under Vasicek specifications. The dashed horizontal line represents the long-term yield  $r^\infty$ .

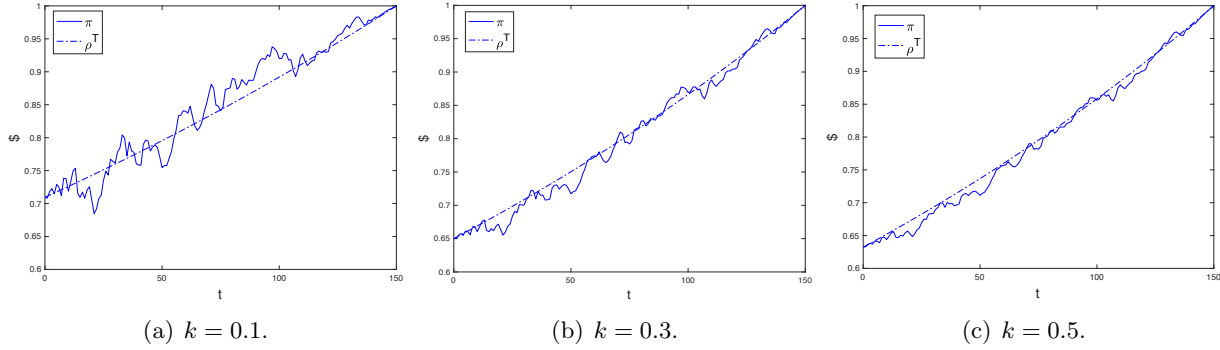


Figure 6: Values of  $\rho_t^T(0, 1_T)$  and of a realization of the no-arbitrage price  $\pi_t(1_T)$  of a zero-coupon bond with fixed maturity  $T$  for any  $t$  in  $[0, T]$ . In all panels, the two prices are represented by a dashed and a solid line, respectively. The employed Vasicek parameters are the ones described in Subsection 6.1.1 except for  $k$  that takes increasing values.

## E Cox, Ingersoll and Ross (1985) dynamics

We enhance the discussion of Subsection 6.1.1 about single-factor affine interest rate models by illustrating our theory when short-term rates follow Cox, Ingersoll, and Ross (1985) dynamics. The coefficients of the instantaneous rate are

$$\mu(t, Y_t) = k\theta - (k - \sigma\xi)Y_t, \quad \sigma(t, Y_t) = \sigma\sqrt{Y_t}$$

with  $k, \theta, \sigma, \xi > 0$  and  $\sigma^2 < 2k\theta$ . Here, the market price of risk is supposed to be  $\xi\sqrt{Y_t}$ . Under the risk-neutral measure  $Q$ , we get the dynamics

$$dY_t = k[\theta - Y_t]dt + \sigma\sqrt{Y_t}dW_t^Q. \quad (30)$$

This specification ensures that interest rates are positive and their volatility raises when they increase. Moreover,  $A(t, T)$  equals

$$\log \left( \left( \frac{2\sqrt{k^2 + 2\sigma^2} \exp\left(\frac{(k + \sqrt{k^2 + 2\sigma^2})(T-t)}{2}\right)}{2\sqrt{k^2 + 2\sigma^2} + (k + \sqrt{k^2 + 2\sigma^2}) \left(\exp\left(\frac{(T-t)\sqrt{k^2 + 2\sigma^2}}{2}\right) - 1\right)} \right)^{2k\theta/\sigma^2} \right)$$

and

$$B(t, T) = \frac{2 \left(\exp\left(\frac{(T-t)\sqrt{k^2 + 2\sigma^2}}{2}\right) - 1\right)}{2\sqrt{k^2 + 2\sigma^2} + (k + \sqrt{k^2 + 2\sigma^2}) \left(\exp\left(\frac{(T-t)\sqrt{k^2 + 2\sigma^2}}{2}\right) - 1\right)}$$

as mentioned in Section 3.2 of Brigo and Mercurio (2006).

Under the forward measure the drift coefficient of the risk-neutral bond price turns out to be  $(1 + \sigma^2 B^2(t, T))Y_t$  and converges a.s. to

$$\left(1 + \sigma^2 \frac{4}{(k + \sqrt{k^2 + 2\sigma^2})^2}\right) Y_t$$

when the trading horizon  $T$  becomes infinitely large. Differently, the drift parameter  $r_0^T$  of the price  $\rho_t^T(0, 1_T)$  converges a.s. to the long-term yield  $r^\infty = k\theta(\sqrt{k^2 + 2\sigma^2} - k)/\sigma^2$ .

We set the parameters  $k = 0.4$ ,  $\theta = 0.04$  and  $\sigma = 0.02$  and we consider a monthly time grid. We also fix  $s = 0$ ,  $\xi = 0.2$  and  $Y_0 = 0.03$ . We obtain a term structures of no-arbitrage prices and yields-to-maturity

qualitatively similar to Figure 5, where a Vasicek model is employed. In the left panel of Figure 7 we fix  $T$  and we consider prices  $\pi_t(1_T)$  and  $\rho_t^T(0, 1_T)$  of pure discount  $T$ -bonds. As discussed in the body of the paper,  $\pi_t(1_T)$  and  $\rho_t^T(0, 1_T)$  are almost indistinguishable in the short-term as well as near maturity.

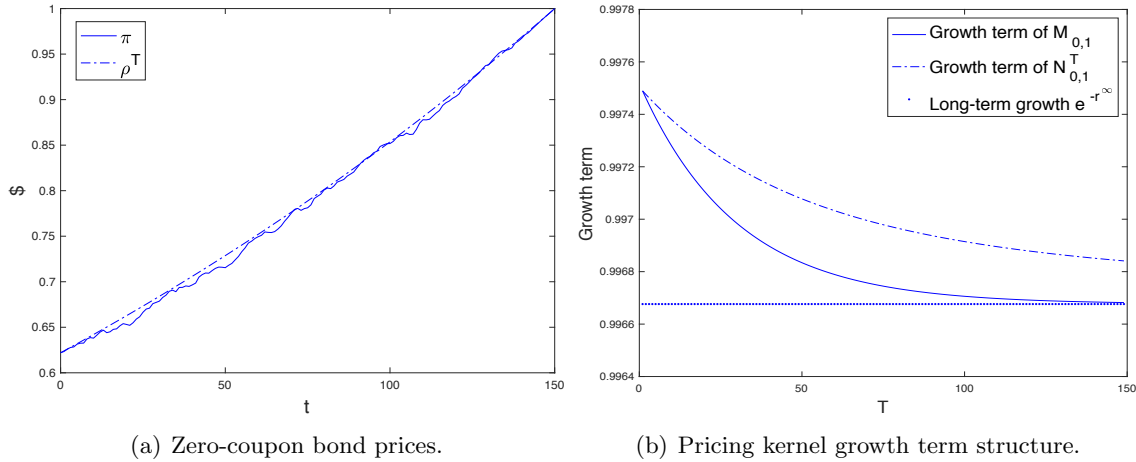


Figure 7: On the left: values of  $\rho_t^T(0, 1_T)$  and of a realization of  $\pi_t(1_T)$  of a pure discount  $T$ -bond for any  $t$  in  $[0, T]$ . The two prices are represented by a dashed and a solid line, respectively. On the right: term structure of growth terms of the pricing kernel  $M_{0,1}$  and the rate-adjusted pricing kernel  $N_{0,1}^T$ . The solid line represents the growth term of  $M_{0,1}$  for increasing horizons  $T$ , while the dashed line regards the growth term of  $N_{0,1}^T$ . The horizontal line is the long-term growth  $e^{-r^\infty}$ . All simulations assume Cox, Ingersoll and Ross dynamics for short rates.

Finally, we illustrate the growth term of the pricing kernel and of the rate-adjusted pricing kernel between time  $s = 0$  and time  $t = 1$ . The right panel of Figure 7 depicts the convergence of both pricing kernel growth terms to  $e^{-r^\infty(t-s)}$  when  $T$  goes to infinity.

## F Application to optimal consumption

In the following we provide a comparison of the consequences of using rate-adjusted valuation for intertemporal consumption choices. We also highlight the role of the long-term yield in providing a threshold for subjective discounting that allows for well-defined consumption policies over infinite horizons.

We consider the complete market of Subsection 6.2. When  $T$  is finite, the market is generated by a risky asset, a pure discount  $T$ -bond and the money market account. When  $T$  is infinite, the zero-coupon bond is replaced by the long bond.

Based on this market, we consider two economies with an agent each. Every agent has subjective discount rate  $\delta$  and time-separable quadratic expected utility on consumption. The first one, that we call *risk-neutral agent*, employs the pricing kernel  $M_{s,t}$  for assessing the budget feasibility of consumption plans. Differently, the second agent, that we call *rate-adjusted*, uses  $N_{s,t}^T$  for discounting future consumption.

We denote by  $w_s$  the  $\mathcal{F}_s$ -measurable initial endowment and by  $c = \{c_t\}_{t \in [s, T]}$  any adapted consumption stream. The risk-neutral consumer maximizes her expected utility among budget-feasible consumption plans:

$$\max_c \mathbb{E}_s \left[ \int_s^T e^{-\delta(\tau-s)} u(c_\tau) d\tau \right] \quad (31)$$

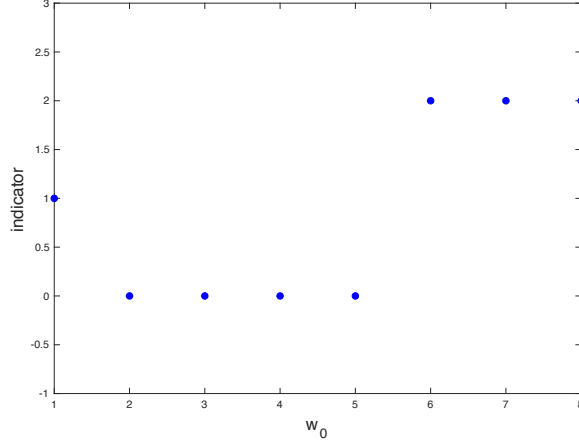


Figure 8: Across increasing initial endowments, the represented function equals 0 when optimal consumption policies are different in the two problems, 1 when optimal consumption policies coincide but do not reach the bliss point and 2 when they are both equal to the bliss point.

$$\text{sub } \mathbb{E}_s \left[ \int_s^T M_{s,\tau} c_\tau d\tau \right] \leq w_s,$$

where  $u(c_\tau) = -(c_\tau - b)^2/2$  and  $b > 0$  is the bliss point. See, e.g. Cochrane (2014). We indicate by  $c^* = \{c_t^*\}_{t \in [s, T]}$  the optimal policy of the risk-neutral agent.

On the contrary, the rate-adjusted consumer employs  $N_{s,t}^T$  for discounting. By choosing among all feasible consumption plans, she solves the optimization problem

$$\max_c \mathbb{E}_s \left[ \int_s^T e^{-\delta(\tau-s)} u(c_\tau) d\tau \right] \quad (32)$$

$$\text{sub } \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T c_\tau d\tau \right] \leq w_s, \quad \mathbb{E}_s \left[ \int_s^T M_{s,\tau} c_\tau d\tau \right] \leq w_s,$$

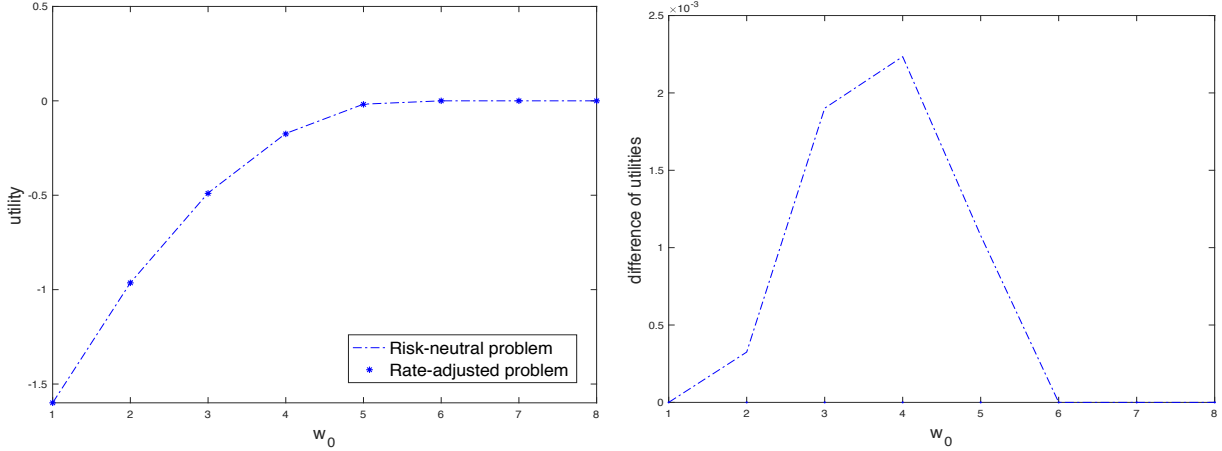
where the rate-adjusted valuation of future consumption streams adds to the budget feasibility constraint of problem (31). We denote the optimal consumption policy of the rate-adjusted agent by  $\hat{c}^*$ .

$c^*$  and  $\hat{c}^*$  may be different consumption plans, depending on the relevance of the rate-adjusted budget feasibility constraint in problem (32). In general, the rate-adjusted agent can never outperform the risk-neutral one because of the larger number of constraints to fulfill. However, there are intervals of the initial endowment in which the two consumers enjoy the same optimal policy. In Figure 8 we consider increasing values of the endowment. We show in simulation when the two optimal consumption streams are distinct and when they coincide. In the latter case, we also individuate the values of initial wealth that allow for optimal policies constantly equal to the bliss point. In all our simulations, we set  $s = 0$  and  $T = 75$ , on monthly basis. For the Vasicek (1977) model we use the parameters  $k = 0.1$ ,  $\theta = 0.04$ ,  $\sigma = 0.01$ ,  $\xi = 0.2$  and  $Y_0 = 0.05$ . Moreover, we set  $\phi = 0.5$ ,  $\eta = 0.2$  and the drift coefficient of the risky asset  $\mu_\tau^X = Y_\tau + 0.05$  for all  $\tau \in [s, T]$ . We fix the subjective discount rate  $\delta = 0.01$ , the bliss point  $b = 1$  and we let the initial endowment vary from 1 to 8.

As a special case, we have that  $c_\tau^* = \widehat{c}_\tau^* = b$  for all  $\tau \in [s, T]$  if

$$w_s \geq b \max \left\{ \mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right], \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] \right\}.$$

In this case, both consumers are very wealthy and they can afford a consumption plan constantly equal to the bliss point over the whole period. For these levels of initial endowment the rate-adjusted budget feasibility constraint turns out to be unnecessary. Moreover, in the previous threshold  $\mathbb{E}_s[\int_s^T N_{s,\tau}^T d\tau] = (1 - e^{-r_s^T(T-s)})/r_s^T$  and this quantity converges to the inverse of  $r^\infty$  when  $T$  goes to infinity.



(a) Maximized expected utilities.

(b) Differences between utilities.

Figure 9: On the left, values of maximized expected utilities in problems (31) and (32) with respect to increasing initial endowments. On the right, difference of maximized utilities across increasing initial endowments.

In Figure 9 we plot the value of the maximized utilities and the difference between the indirect utilities in the two economies for increasing initial wealth. In our simulation, the difference between indirect utilities, when present, is tiny.

We provide the exact solutions of the risk-neutral and the rate-adjusted consumption problems in the next propositions.

**Proposition 13 (Risk-neutral problem)** *The solution of problem (31) is the following. For all  $\tau \in [s, T]$ ,*

- i) if  $w_s \geq b\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$ , then  $c_\tau^* = b$ ;
- ii) if  $w_s < b\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$ , then  $c_\tau^* = b - e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$  with

$$\lambda_s = \frac{b\mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] - w_s}{\mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right]}.$$

**Proof of Proposition 13.** The Lagrangian function associated with problem (31) is

$$\mathcal{L} = \mathbb{E}_s \left[ \int_s^T e^{-\delta(\tau-s)} u(c_\tau) d\tau \right] - \lambda_s \left( \mathbb{E}_s \left[ \int_s^T M_{s,\tau} c_\tau d\tau \right] - w_s \right),$$

where  $\lambda_s \in L^0(\mathcal{F}_s)$  is non-negative. By differentiating with respect to any realization of  $c_\tau$ , we get the first order condition:  $u'(c_\tau) = e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$ . Since  $(u')^{-1}(y) = b - y$ , we obtain  $c_\tau^* = b - e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$ .

If  $\lambda_s = 0$ , then  $c_\tau^* = b$  for all  $\tau \in [s, T]$  and the constraint delivers the condition  $w_s \geq b \mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$ . Hence, i) is proved.

If  $\lambda_s > 0$ , the constraint is binding and so  $\lambda_s$  takes the expression in ii). ■

**Proposition 14 (Rate-adjusted problem)** *The solution of problem (32) is the following. For all  $\tau \in [s, T]$ ,*

i) if  $w_s \geq b \max\{\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau], \mathbb{E}_s[\int_s^T N_{s,\tau}^T d\tau]\}$ , then  $\hat{c}_\tau^* = b$ ;

ii) if

$$\begin{aligned} & \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right) w_s \\ & < b \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] \right. \\ & \quad \left. - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] \right) \end{aligned}$$

and

$$\begin{aligned} & \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right) w_s \\ & < b \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] \right. \\ & \quad \left. - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] \right), \end{aligned}$$

then  $\hat{c}_\tau^* = b - e^{\delta(\tau-s)}(\lambda_s M_{s,\tau} + \eta_s N_{s,\tau}^T)$  with

$$\begin{aligned} \lambda_s = \xi & \left\{ b \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] \right. \right. \\ & \quad \left. \left. - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] \right) \right. \\ & \quad \left. - w_s \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right) \right\}, \end{aligned}$$

$$\begin{aligned} \eta_s = \xi & \left\{ b \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] \right. \right. \\ & \quad \left. \left. - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] \right) \right. \\ & \quad \left. - w_s \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right) \right\}, \end{aligned}$$

$$\begin{aligned} \xi^{-1} &= \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] \\ &\quad - \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right)^2. \end{aligned}$$

iii) if  $w_s < b\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$  and

$$\begin{aligned} &\left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right) w_s \\ &\geq b \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T \right] \right. \\ &\quad \left. - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \mathbb{E}_s \left[ \int_s^T M_{s,\tau} \right] \right), \end{aligned}$$

then  $\widehat{c}_\tau^* = b - e^{\delta(\tau-s)} \lambda_s M_{s,\tau}$  with

$$\lambda_s = \frac{b\mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] - w_s}{\mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right]};$$

iv) if  $w_s < b\mathbb{E}_s[\int_s^T N_{s,\tau}^T d\tau]$  and

$$\begin{aligned} &\left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \right) w_s \\ &\geq b \left( \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] \mathbb{E}_s \left[ \int_s^T M_{s,\tau} \right] \right. \\ &\quad \left. - \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T \right] \right), \end{aligned}$$

then  $\widehat{c}_\tau^* = b - e^{\delta(\tau-s)} \eta_s N_{s,\tau}^T$  with

$$\eta_s = \frac{b\mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] - w_s}{\mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right]}.$$

**Proof of Proposition 14.** The Lagrangian function related to problem (32) is

$$\begin{aligned} \mathcal{L} &= \mathbb{E}_s \left[ \int_s^T e^{-\delta(\tau-s)} u(c_\tau) d\tau \right] - \lambda_s \left( \mathbb{E}_s \left[ \int_s^T M_{s,\tau} c_\tau d\tau \right] - w_s \right) \\ &\quad - \eta_s \left( \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T c_\tau d\tau \right] - w_s \right) \end{aligned}$$

with non-negative  $\lambda_s, \eta_s \in L^0(\mathcal{F}_s)$ . Here, the first order condition is  $u'(c_\tau) = e^{\delta(\tau-s)}(\lambda_s M_{s,\tau} + \eta_s N_{s,\tau}^T)$  and so  $c_\tau^* = b - e^{\delta(\tau-s)}(\lambda_s M_{s,\tau} + \eta_s N_{s,\tau}^T)$ .

Case i) obtains when  $\lambda_s = \eta_s = 0$ . In this case,  $c_\tau^* = b$  for all  $\tau \in [s, T]$ .

In ii)  $\lambda_s$  and  $\eta_s$  are positive and so their constraints are binding. Therefore, a linear system for  $\lambda_s$  and  $\eta_s$  arises:

$$\begin{cases} \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau \right] \lambda_s + \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \eta_s & = b \mathbb{E}_s \left[ \int_s^T M_{s,\tau} d\tau \right] - w_s \\ \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau \right] \lambda_s + \mathbb{E}_s \left[ \int_s^T e^{\delta(\tau-s)} (N_{s,\tau}^T)^2 d\tau \right] \eta_s & = b \mathbb{E}_s \left[ \int_s^T N_{s,\tau}^T d\tau \right] - w_s. \end{cases}$$

The unique solution is the one in the statement of ii). The conditions on the parameters in the claim ensure the positivity of  $\lambda_s$  and  $\eta_s$ .

In case iii),  $\lambda_s$  is positive, while  $\eta_s = 0$ . Thus  $c_\tau^* = b - e^{\delta(\tau-s)} \lambda_s M_{s,\tau}$ . The feasibility constraint with  $M_{s,\tau}$  is binding and so we can derive  $\lambda_s$  as in the statement. The constraint with  $N_{s,\tau}^T$  delivers the inequality in the claim.

Case iv) is specular to iii). Just replace  $M_{s,\tau}$  with  $N_{s,\tau}^T$  and conversely. ■

In general, the ex-ante expected difference between the risk-neutral optimal policy and the rate-adjusted one is given by

$$\mathbb{E}_s [c_\tau^* - \widehat{c}_\tau^*] = \alpha_s e^{-(r_s^T - \delta)(\tau-s)} - \beta_s e^{-(r_s^T - \delta)(\tau-s)}$$

for some  $\alpha_s, \beta_s \in L^0(\mathcal{F}_s)$ . The difference depends on the relations between the yield  $r_s^T$ , the time-dependent yield  $r_s^T$  and the subjective discount rate  $\delta$ . In Figure 10 we consider a value of the endowment that delivers different optimal consumption streams, specifically  $w_0 = 2$ . Then, we depict the two expected optimal consumption plans and their difference, which turn out to be very small in our simulations.

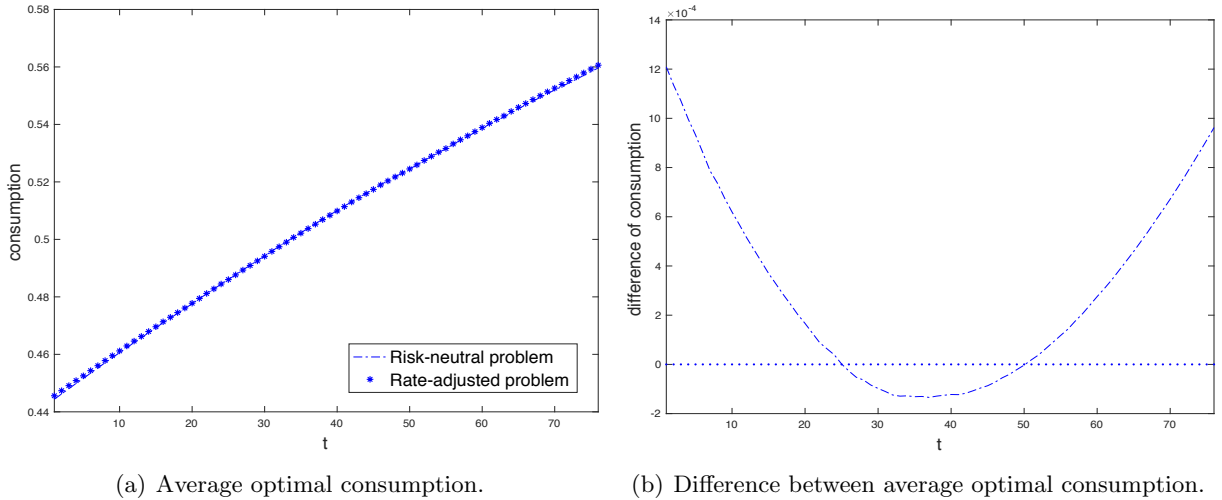


Figure 10: On the left, average optimal consumption plans over time for the initial endowment  $w_0 = 2$  that delivers different optimal policies in problems (31) and (32). On the right, difference between the two average consumption policies.

When the horizon is infinite, the rate-adjusted agent employs  $N^\infty$  as pricing kernel. Moreover, the relation between the long-term yield  $r^\infty$  and the subjective discount rate  $\delta$  comes into play when we analyze the asymptotic behavior of expected consumption over time.

On the one hand, if  $\delta < r^\infty$  both  $\mathbb{E}_s[c_\tau^*]$  and  $\mathbb{E}_s[\widehat{c}_\tau^*]$  converge to the bliss point. The fact that the subjective discounting does not exceed the long-term yield constitutes a transversality condition that rephrases Corollary 1 in Ross (2015).



On the other hand, if  $\delta > r^\infty$  both  $\mathbb{E}_s[c_\tau^*]$  and  $\mathbb{E}_s[\widehat{c}_\tau^*]$  fall. In this case of impatience, no finite optimal consumption plan can be found.

In general, the convergence rates of  $c_\tau^*$  and  $\widehat{c}_\tau^*$  satisfy the asymptotic relation

$$\frac{\log \mathbb{E}_s [b - c_\tau^*] - \log \mathbb{E}_s [b - \widehat{c}_\tau^*]}{\tau - s} \xrightarrow{P} 0, \quad \tau \rightarrow +\infty,$$

that employs the distances of optimal policies from the bliss point.

Summing up, optimal consumption plans  $c^*$  and  $\widehat{c}^*$  can be different but:

- when initial endowment is high enough,  $c^*$  and  $\widehat{c}^*$  coincide;
- when the horizon is infinite and  $\delta < r^\infty$ ,  $\mathbb{E}_s[c_\tau^*]$  and  $\mathbb{E}_s[\widehat{c}_\tau^*]$  coincide asymptotically for large  $\tau$ .

Therefore, we can claim that expected consumption plans may differ just in the short term. Moreover, the long-term yield provides a threshold for the subjective discount rate in the long run.

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