Long-term risk with stochastic interest rates*

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Abstract

Investors with heterogeneous trading horizons require compensation for the exposure to different risks. The no-arbitrage valuation over increasing horizons is described by the evolution of stochastic discount factors (SDFs). Each of them exhibits a multiplicative decomposition into deterministic growth term, permanent and transient component, provided by Hansen and Scheinkman (2009). In particular, the growth rate captures the deterministic discounting for risks that are relevant in the long term. When interest rates in the market are constant, the SDF growth rate coincides with the instantaneous rate. On the contrary, when rates of interest are stochastic, the SDF growth rate is given by the long-term yield of zero-coupon bonds, which is unsuitable for instantaneous no-arbitrage valuation.

We show how to reconcile the long-run properties of the SDF with the instantaneous relations between returns and rates in stochastic-rate markets. In particular, we introduce a rate adjustment in pricing that isolates the short-term variability of rates. No-arbitrage prices are then factorized into rate-adjusted prices and a rate adjustment that is absent when interest rates are constant. Rate-adjusted prices employ constant yields to maturity for discounting future payoffs and constitute indifference prices for investors that neglect the term structure of rates. The rate-adjusted SDF features the same long-term growth rate of the SDF in the market but has no transient component in its Hansen-Scheinkman decomposition. Therefore, rate-adjusted prices provide the proper valuation for long-term interest rate risk and shed light on the impact of temporary and long-lasting interest rate shocks on security prices.

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1 Motivations and main results

A deep analysis of the term structure of interest rates is essential for dynamic asset pricing theory. Once fixed-income securities are issued by an institution, they are likely to display time-varying yields to maturity in the secondary market. In fact, different yields embody the changeable expectations of market participants about future events that could impinge on the credit market at different horizons. Monetary policy interventions may directly affect short-term rates while long-term maturities are sensitive to firms investment decisions, households attitudes and output growth, among the others. Moreover, the relation between short-term and long-term bond yields is an indicator of the overall performance of the economy. For instance, inverted yield curves may constitute a predictor for recessions (see Ang, Piazzesi, and Wei, 2006).

Understanding the consistent aggregation of interest rate risks associated with heterogeneous maturities has been a challenge since the last century. A substantial stream of economic literature involves expectation theory, introduced by Fisher (1930) and developed by Keynes (1930). Subsequent elaborations are due to Lutz (1940) and Hicks (1946), and alternative explanations are provided by Culbertson (1957) and Modigliani and Sutch (1966). See the surveys by Cox, Ingersoll, and Ross (1985) and Russell (1992).

Similarly, in dynamic asset pricing theory, investors with different trading horizons require compensation for the exposures to sources of randomness with heterogeneous duration. As a consequence, long-term premia are the outcome of the intertemporal aggregation of local risk exposures. Importantly, the components of economic variables that survive in the long run are free from inaccuracies due to temporary risk adjustments and are likely to reflect economic fundamentals. A seminal paper in this area is Bansal and Yaron (2004), which shows how a small (but persistent) perturbation on consumption and dividend growth can affect long-term returns, and proposes a potential explanation to the equity premium puzzle of Mehra and Prescott (1985). Later, Hansen, Heaton, and Li (2008) provide a theoretical analysis of risk-return trade-offs across increasing maturities.

Hansen and Scheinkman (2009) develop a general operator framework to connect short- and long-run risk compensations. Their valuation of cashflows exploits the evolution of both stochastic growth and stochastic discount functionals. In particular, in arbitrage-free markets every stochastic discount factor features a multiplicative decomposition in a deterministic growth term, a permanent (or martingale) component and a transient factor.

The martingale component induces a probability measure change in the long run and is responsible for a large variance explanation. In equilibrium this decomposition impacts the asymptotic dynamics of aggregate consumption and wealth through investors' marginal utility, as assessed by Alvarez and Jermann (2005) and Hansen (2012).

Stochastic discount factors reflect the risk-neutral valuation of state-contingent payoffs in viable markets. In particular, the deterministic growth rate elicited by the Hansen-Scheinkman decomposition provides a proper discounting adjusted for risks that are relevant in the long run. When interest rates in the market are constant over time, the pricing kernel growth rate coincides with the same instantaneous rate of interest at any maturity under scrutiny. When rates are stochastic, their variability is captured by the interplay of stochastic discounting and stochastic growth, as discussed in Borovička, Hansen, Hendricks, and Scheinkman (2011). Unfortunately, in this case it is hard to achieve a synthetic characterization of pricing kernel growth rates across increasing horizons.

This work fills this gap and builds a bridge between fixed-income and long-term risk literatures, showing the extent to which the intertemporal aggregation of instantaneous random rates contributes to pricing kernel growth rates at increasingly large horizons. Moreover, by shifting the maturity farther and farther over time, we show how to retrieve the asymptotic deterministic growth rate inferred by Qin and Linetsky (2017). Notably, our results do not depend on the specific dynamics of interest rates.

1.1 Main results

We consider a continuous-time arbitrage-free market in the time interval $[0, T]$, composed of both risky and fixed-income securities that depend on a stochastic instantaneous rate Y_t. We concentrate on price dynamics, while simultaneously focusing on the evolution of stochastic discount factors. We develop the theory in a conditional setting.

Regarding price dynamics, we start considering the no-arbitrage price $\pi_t(h_T)$ at time $0 \leqslant t \leqslant T$ of an attainable payoff with maturity T. When rates of interest are constant over time, the no-arbitrage condition leads to the equality

$$\text{instantaneous asset return} = \text{instantaneous risk-free rate}$$

under a risk-neutral measure Q. In the language of Hansen and Scheinkman (2009) and Hansen (2012), this relation is formalized by the eigenvalue-eigenvector problem

$$\mathcal{A}\pi_t = r\ \pi_t, \qquad t \in [0, T]. \tag{1}$$

Here, $\mathcal{A}$ is the extended infinitesimal generator of the Markov price process. $\mathcal{A}$ works as a differential operator and generalizes the ordinary differential equation satisfied by risk-free bonds with interest rate r. Importantly, the eigenvalue r has a prominent role in the evolution of the stochastic discount factor since it captures its deterministic growth rate. Indeed, the pricing kernel in any time interval $[s,t]$ is $M_{s,t} = e^{-r(t-s)}L_{s,t}$, where $L_{s,t}$ is the conditional Radon-Nikodym density of Q with respect to the physical probability P.

When rates are constant over time, the instantaneous rate determines both the risk-neutral price dynamics and the pricing kernel growth rate, irrespective of the time interval under consideration. In a stochastic-rate setting, the action of instantaneous rates is more convoluted. Indeed, while eq. (1) may be rephrased at any instant t by using the random rate Y_t instead of r, the sole instantaneous rate Y_t is unable to subsume the stochastic discount factor growth rate on any given time period. In particular, the pricing kernel in $[s,t]$ takes the form $M_{s,t} = e^{-\int_s^t Y_\tau d\tau}L_{s,t}$.

Comparing the expressions of $M_{s,t}$ in the two contexts, we obtain an indication of the heavy measurability requirement of the growth term $e^{-\int_s^t Y_\tau d\tau}$ with respect to the deterministic $e^{-r(t-s)}$. The definition of a growth rate for $M_{s,t}$ is indeed challenging when rates of interest are floating over time. Qin and Linetsky (2017) individuate a deterministic *long-term yield* as the asymptotic growth rate of the stochastic discount factor. Nevertheless, the quantity identified by Qin and Linetsky – which is the limit of bonds yields at infinite horizons – is not suitable for characterizing instantaneous returns in the sense of eq. (1). The conceptual reason beyond these difficulties rests upon the dual nature of stochastic interest rates. On the one hand, they represent a proxy for the (absent) risk-free rates. On the other, they constitute a source of randomness per se.

The main contribution of this paper is to provide a generalization of eq. (1) for stochastic-rate settings where the employed eigenvalue is determinant for the pricing kernel growth rate. The previously described relations for rates that are constant over time follow as a special case of our construction. In addition, our generalization is consistent with the asymptotic results of the long-term risk literature.

Our theory is based on the introduction of *rate-adjusted prices* that provide hedging from interest rate variability. In general, given an attainable payoff h_T at time T, we denote by $\pi_t(h_T)$ its no-arbitrage price at t. For example, $\pi_t(1_T)$ is the no-arbitrage price of a pure discount T-bond. The rate-adjusted price of h_T at time t in the interval $[s,T]$, denoted by $\rho_t^T(s,h_T)$, satisfies

$$\underbrace{\rho_t^T\left(s,h_T\right)}_{rate-adjusted\ price} = \underbrace{e^{r_s^T(t-s)}\frac{\pi_s\left(1_T\right)}{\pi_t\left(1_T\right)}}_{adjustment} \cdot \underbrace{\pi_t\left(h_T\right)}_{no\ arbitrage\ price},$$

where r_s^T is the yield-to-maturity of a zero-coupon T-bond at time s. If rates of interest are constant, rate-adjusted and no-arbitrage prices coincide. In general, the two prices are equal at the instants s and T.

Interestingly, ρ^T can be interpreted as an indifference price for an investor that ignores the variability of rates, or as the conversion price of a convertible bond. In particular, the rate-adjusted price of a zero-coupon bond is the no-arbitrage price of the same security in a parallel market with constant interest rates equal to the yield r_s^T. Moreover, when the horizon T goes to infinity, the ratio between ρ_t^T and π_t is convergent both in probability and in expectation.

To look at the matter from the standpoint of stochastic discount factors, we define the *rate-adjusted pricing kernel* in the interval $[s,t]$ with $s \leqslant t \leqslant T$ by

$$N_{s,t}^T = e^{-r_s^T(t-s)} \frac{\pi_t(1_T)}{\pi_s(1_T)} M_{s,t}.$$

As expected, $N_{s,t}^T$ and $M_{s,t}$ coincide when interest rates are constant.

To formalize our differential pricing relations in a conditional setting, we introduce a mathematical instrument that allows us to differentiate stochastic processes on the time window $[s,T]$ by disentangling the role of known information at instant s: the weak time-derivative in $[s,T]$, denoted by $\mathcal{D}$. This tool is a generalization of the weak time-derivative of Marinacci and Severino (2018) that applies to a wide class of semimartingale processes. Similarly to the original notion, the weak time-derivative in $[s,T]$ provides a handy characterization of conditional martingales and so it is useful for dealing with discounted no-arbitrage prices and forward prices. These processes feature, indeed, null weak time-derivative in $[s,T]$.

Our main results are Theorems 2 and 9, in which we prove that rate-adjusted prices and rate-adjusted pricing kernels satisfy the differential relations

$$\mathcal{D}\rho_t^T = r_s^T \rho_t^T, \qquad \mathcal{D}N_{s,t}^T = -r_s^T N_{s,t}^T \tag{2}$$

for any t in $[s,T]$, where the first equality holds under the forward measure and the second one under the physical measure. These equations parallel the relations satisfied by no-arbitrage prices and by the pricing kernel $M_{s,t}$ when interest rates are constant:

$$\mathcal{D}\pi_t = r\pi_t, \qquad \mathcal{D}M_{s,t} = -rM_{s,t},$$

where the two equalities hold under the risk-neutral measure Q and the measure P, respectively. The equations on the left rephrase the relation between returns and rates, while the parameters in the right-hand side equalities identify the pricing kernel growth rates.

The constant rate is replaced by the yield r_s^T, conveying the intuition that pure discount bonds play the role of risk-free assets in a context in which the money market account is unforeseeable. The results are robust when the horizon is moved to infinity, as we discuss in Subsection 3.6. In this case, the long-term yield arises as the growth rate for both the rate-adjusted pricing kernel and $M_{s,t}$. Indeed, the long-term rate-adjusted pricing kernel $N^{\infty}_{s,t}$ differs from $M_{s,t}$ just in the transient component in the Hansen and Scheinkman (2009) decomposition. Such component is trivial only for $N^{\infty}_{s,t}$ when interest rates are stochastic. This property is key to build our generalization and puts asset pricing with either constant or fluctuating rates into the same perspective.

The dynamics of eq. (2) are conditional on the information structure available at date s. We are actually considering an eigenvalue-eigenvector problem in which the eigenvalue is a random variable, known at the beginning of the trading interval. This approach combines the intuition of conditional asset pricing theory – that dates back to Hansen and Richard (1987) – with the Perron-Frobenius theory applied to economic theory by Hansen and Scheinkman (2009).

As we discuss in the body of the paper, results similar to eq. (2) are unlikely to hold for no-arbitrage prices when rates are stochastic: the instantaneous rate Y_t cannot serve as eigenvalue and the transitory component of $M_{s,t}$ hinders the differentiation. Nonetheless, rate-adjusted prices are able to reach the purpose. Indeed, pure discount bond yields play both the role of eigenvalues (for rate-adjusted prices) and that of growth rates (for rate-adjusted pricing kernels), generalizing the features of the constant-rate case. Table 1 at the end of Section 4 summarizes our results.

The paper is organized as follows. Section 2 introduces the no-arbitrage pricing framework, the assumptions for long-term convergence and the weak time-derivative in $[s, T]$. Section 3 describes the properties of rate-adjusted prices and their instantaneous relations. Section 4 is devoted to the pricing kernel growth rates. In Section 5 we illustrate our results in affine market models. The Appendix discusses several theoretical issues and provides all the proofs, additional simulations and an application of rate-adjusted valuation to optimal consumption problems.

2 Asset pricing framework

We start with describing a general continuous-time arbitrage-free market with stochastic interest rates. We then focus on the conditions that ensure long-term convergences, we describe the forward measures and we discuss the main properties of weak time-derivatives. All technical details are collected in Appendix A.

2.1 Arbitrage-free market

We fix $T > 0$ and consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and is left-continuous in T. We build a financial market by considering two adapted random processes $X = \{X_t\}_{t\in[0,T]}$ and $Y = \{Y_t\}_{t\in[0,T]}$. The process X is N-dimensional and consists of prices of N primary risky assets that generate the market, i.e. $X_t = [X_t^1, \ldots, X_t^N]'$ for all t. The one-dimensional process Y represents the instantaneous interest rate. By computing the integral $\int_0^T Y_\tau d\tau$ pathwise, we are able to define the *money market account* process $\{e^{\int_0^t Y_\tau d\tau}\}_{t\in[0,T]}$. Moreover, pure discount bonds with any possible maturity are traded, too. For simplicity the face value of these instruments is one unit.

At any instant t we define the vector of relative prices $Z_t = e^{-\int_0^t Y_\tau d\tau} X_t$ obtained by discounting asset prices by the value of the money market account. We assume that our price system satisfies the *no free lunch with vanishing risk* condition and that relative asset prices are semimartingales. Therefore, by the First Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer (1998) there exists a probability measure equivalent to P such that Z is a *sigma-martingale*. We assume that at least one of these sigma-martingale measures, denoted by Q, is actually an *equivalent martingale measure*. We indicate by L_T the Radon-Nikodym derivative of Q with respect to P and we set $L_t = \mathbb{E}_t[L_T]$ for all $t \in [0,T]$, where $\mathbb{E}_t$ is the conditional expectation with respect to $\mathcal{F}_t$ under P. We also define $L_{t,T} = L_T/L_t$.

We denote by $S = \{S_t\}_{t\in[0,T]}$ the strictly positive *stochastic discount factor* process associated with the measure Q, namely $S_t = e^{-\int_0^t Y_\tau d\tau} L_t$ for all t. In addition, we define the pricing kernel in any time interval $[s,t]$ with $s \leqslant t \leqslant T$ by $M_{s,t} = S_t/S_s = e^{-\int_s^t Y_\tau d\tau} L_{s,t}$.

The no-arbitrage price at time t of a zero-coupon bond with redemption date T is $\pi_t(1_T)$ and the related yield to maturity is $r_t^T = -\log \pi_t(1_T)/(T-t)$. We also define r_T^T as the a.s. limit of r_t^T when t approaches T. The yield r_t^T may be interpreted as an average rate of interest which is ex-ante equivalent to the compounding of all forthcoming instantaneous rates Y_τ when τ spans the interval $[t,T]$. It is also referred to as *internal rate of return*. Clearly, if interest rates are constant over time (and deterministic), the yield coincides with the instantaneous rate.

2.2 Long-term assumptions

Our semimartingale framework is compatible with the setting of Qin and Linetsky (2017) that formalize the convergence of bond yields, forward measures and stochastic discount factors when the horizon T becomes larger and larger. Borrowing from them, we assume that the stochastic discount factor S_t is a strictly positive semimartingale such that $\mathbb{E}[S_T/S_t]$ is finite for all $0 \leqslant t < T$. Moreover, we assume that, for all $t > 0$, when T goes to infinity, $\mathbb{E}_t[S_T]/\mathbb{E}[S_T]$ converges in L^1 to a positive $\mathcal{F}_t$-measurable random variable G_t^∞.

Under these assumptions, for any t, the yield to maturity r_t^T converges in probability to a positive deterministic *long-term yield* r^∞. This result is consistent with the persistence of the yield curve over large maturities widely documented empirically and discussed, for instance, in Diebold and Li (2006). Notably, r^∞ is not dependent on t, consistently with the impossibility of falling long-term rates illustrated by Dybvig, Ingersoll Jr, and Ross (1996) and Hubalek, Klein, and Teichmayn (2002).

The long-term yield is associated with a long bond, obtained by a roll-over portfolio over zero-coupon bonds across increasing maturities. The time t value of this portfolio is denoted by B_t^∞, while b_t^∞ is the related long-term discounted value, namely $b_t^\infty = e^{-r^\infty t} B_t^\infty$.

2.3 Forward measures

By using as numéraire the no-arbitrage price of a zero-coupon T-bond, we construct the forward measure with horizon T or, simply, *T-forward measure* and we denote it by F^T. See e.g. Geman, El Karoui, and Rochet (1995). This probability measure is equivalent to Q and we indicate its Radon-Nikodym derivative with respect to P by G_T^T. Moreover, we set $G_t^T = \mathbb{E}_t[G_T^T]$ for any $t \in [0,T]$ and we define $G_{t,T}^T = G_T^T / G_t^T$. In the special case in which rates of interest are constant over time, F^T coincides with Q.

Using G_t^T, the stochastic discount factor and the pricing kernel in any interval $[s,t]$ may be expressed as

$$S_t = e^{r_t^T (T-t) - r_0^T T} G_t^T, \qquad M_{s,t} = e^{r_t^T (T-t) - r_s^T (T-s)} G_{s,t}^T.$$

Although $M_{s,t}$ refers to the time window $[s,t]$, its expression depends on the ultimate horizon T through the density of the T-forward measure.

In our setting the T-forward measure provides a handy representation of risk-neutral asset prices. Consider, for instance, an attainable $\mathcal{F}_T$-measurable payoff h_T. Its no-arbitrage price at any intermediate time t can be written in equivalent ways, depending on the numéraire change:

$$\pi_t(h_T) = \mathbb{E}_t^Q \left[e^{-\int_t^T Y_\tau d\tau} h_T \right] = e^{-r_t^T (T-t)} \mathbb{E}_t^{F^T} [h_T]. \tag{3}$$

The right-hand side of this expression actually makes a fruitful bridge between constant-rate and stochastic-rate asset valuation.

We now consider the case in which the horizon T becomes arbitrarily large. Under the previous regularity assumptions, Qin and Linetsky (2017) prove that F^T strongly converges to the *long-term forward measure* F^∞ when T goes to infinity. At any time t the Radon-Nikodym derivative of F^∞ with respect to P is G_t^∞ and the collection of all G_t^∞ constitutes a P-martingale process. In addition, we set $G_{s,t}^\infty = G_t^\infty / G_s^\infty$ for all s and t. Interestingly, the long-term forward measure is related to a specific numéraire that is exactly the price B_t^∞ of the long bond.

Finally, Qin and Linetsky (2017) prove that the pricing kernel $M_{s,t}$ satisfies the long-run decomposition

$$M_{s,t} = e^{-r^\infty (t-s)} \, \frac{b_s^\infty}{b_t^\infty} \, G_{s,t}^\infty. \tag{4}$$

In a Markovian environment, G_t^∞ / G_s^∞ defines the *permanent* (or *martingale*) *component* of Hansen and Scheinkman (2009) decomposition, while b_s^∞ / b_t^∞ constitutes the *transient component* and r^∞ is the deterministic long-term growth rate. For the empirical estimation of these terms see Christensen (2017) and Qin, Linetsky, and Nie (2018). A non-Markovian example is provided by Qin and Linetsky (2018). Furthermore, from eq. (4) it is easy to write the no-arbitrage price of any payoff h_T by using the long-term forward measure:

$$\pi_t \left(h_T \right) = e^{-r^\infty (T-t)} \mathbb{E}_t^{F^\infty} \left[\frac{b_t^\infty}{b_T^\infty} h_T \right]. \tag{5}$$

2.4 Weak time-derivative in $[s, T]$

In order to analyze the increments of stochastic processes we need a differential operator. Marinacci and Severino (2018) introduce the *weak time-derivative* that applies to a wide class of semimartingales and generalizes the infinitesimal generator for Markov processes. Here we extend this notion to a conditional setting. Appendix A collects the most technical details.

We fix an instant $s \in [0, T]$ and we consider the conditional space $L_s^1(\mathcal{F}_T)$ composed of variables $f \in L^0(\mathcal{F}_T)$ such that $\mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)$ and endowed with the L^0-valued metric $d(f, g) = \mathbb{E}_s[|f - g|]$. Moreover, we denote by $\mathcal{U}_s$ the L^0-module of adapted processes $u : [s, T] \to L_s^1(\mathcal{F}_T)$ that are L_s^1-right-continuous in $[s, T)$ and L_s^1-left-continuous in T.

We then define the weak time-derivative for processes in $\mathcal{U}_s$. For any $t \in [s, T]$ the definition uses the space $C_c^1((t, T), L^0(\mathcal{F}_s))$ of functions $\varphi_s : [t, T] \to L^0(\mathcal{F}_s)$ that have compact support in (t, T) and are continuously differentiable over time.

Definition 1 *We say that a process $u \in \mathcal{U}_s$ is* weakly time-differentiable in $[s, T]$ *when there exists a process $v \in \mathcal{U}_s$ such that for every $t \in [s, T]$*

$$\int_t^T \mathbb{E}_s \left[v_\tau \mathbf{1}_{A_t} \right] \varphi_s(\tau) d\tau = - \int_t^T \mathbb{E}_s \left[u_\tau \mathbf{1}_{A_t} \right] \varphi_s'(\tau) d\tau$$

for all $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t, T), L^0(\mathcal{F}_s))$. In this case, we call v a weak time-derivative *of u in $[s, T]$.*

The integrals in Definition 1 are pathwise integrals of processes in $L^0(\mathcal{F}_s)$. In fact, Definition 1 generalizes the weak time-derivative of Marinacci and Severino (2018), where $s = 0$ and deterministic test functions are employed.

The weak time-derivative in $[s, T]$ is unique (see Proposition 11 in Appendix A) and we denote it by $\mathcal{D}u$. Then, we introduce the L^0-submodules of $\mathcal{U}_s$

$$\mathcal{U}_s^1 = \{u \in \mathcal{U}_s : \ u \text{ is weakly time} - \text{differentiable in } [s, T]\},$$

$$\mathcal{U}_s^\infty = \{u \in \mathcal{U}_s : \ u \text{ is infinitely weakly time} - \text{differentiable in } [s, T]\}.$$

$\mathcal{F}_s$-measurable functions play the role of multiplicative constants for the weak time-derivative in $[s, T]$. Indeed, given a process $u \in \mathcal{U}_s^1$ and $\xi_s \in L^0(\mathcal{F}_s)$, the process defined by $\xi_s u_t$ for all $t \in [s, T]$ belongs to $\mathcal{U}_s^1$, too, and

$$\mathcal{D}(\xi_s u) = \xi_s \ \mathcal{D}u. \tag{6}$$

This property allows us to deal with $\mathcal{F}_s$-measurable parameters in the differential equations and to study eigenvalue-eigenvector problems for $\mathcal{D}$ with $\mathcal{F}_s$-measurable eigenvalues.

The key feature of weak time-derivatives in $[s, T]$ is the characterization of *conditional* (or *generalized*) *martingales*. By this terminology we mean processes u defined in the time interval $[s, T]$ with all the properties of martingales except for integrability, which is replaced by the weaker condition $\mathbb{E}_t[u_\tau] \in L^0(\mathcal{F}_t)$ for all $s \leqslant t \leqslant \tau \leqslant T$. See, for instance, Chapter VII, §1 of Shiryaev (1996). Importantly, a process belongs to $\mathcal{U}_s^1$ and has null weak time-derivative in $[s, T]$ if and only if it is a conditional martingale, as proved in Proposition 12 in Appendix A.

When the risk-neutral probability Q is considered, a simple example of conditional martingale is provided by the process of *future prices* of a claim with expiry T at any t in $[s, T]$. Instead, under the forward measure, *forward prices* in $[s, T]$ for the settlement date T are conditional martingales. See, for instance, Sections 9.6 and 11.5 in Musiela and Rutkowski (2005). From our perspective, both these processes exhibit null weak time-derivatives in $[s, T]$ with respect to different measures.

The weak time-derivative in $[s, T]$ is also useful for the analysis of no-arbitrage prices because, after discounting by the money-market account, they are martingales under Q. Moreover, the weak time-derivative captures the drift of semimartingale processes and applies to a wide range of continuous-time asset pricing models. See, for instance, Marinacci and Severino (2018) and the example in Subsection 3.1.

3 Pricing equation

We formulate and solve a *rate-adjusted* pricing equation for the valuation of random payments in a market with stochastic interest rates. We, then, compare the solution of the equation with the usual risk-neutral pricing formula. We interpret rate-adjusted prices as indifference prices (that hedge from interest rate randomness) and as conversion prices for hybrid securities. We finally study their properties in the long run.

3.1 Rate-adjusted and no-arbitrage pricing

In the time interval $[s, T]$ we face the problem of evaluating an $\mathcal{F}_T$-measurable payoff h_T. As illustrated in Subsection 2.3, we can express the time t no-arbitrage price of h_T by using both the measure Q and the forward measure. Up to replacing the risk-neutral measure with F^T and instantaneous rates with bond yields, the right-hand side of eq. (3) formally traces no-arbitrage pricing with constant rates of interest.

As discussed by Hansen and Scheinkman (2009), in an arbitrage-free market with constant rate of interest r, asset prices satisfy the relation

$$\mathcal{A}\pi = r\ \pi, \tag{7}$$

where $\mathcal{A}$ is the extended generator of an underlying Markov process and the risk-neutral measure is employed. This eigenvalue-eigenvector problem has its roots in the Perron-Frobenius theory and captures the essence of no-arbitrage. Indeed, it relates infinitesimal price increments of a possibly risky security with the interest rate deriving from a locally riskless investment. This approach is known in the financial literature since Cox and Ross (1976) derivation of Black-Scholes formula via a hedging portfolio. Moreover, the rate r, up to a sign change, defines the growth rate of the stochastic discount factor in the celebrated Hansen-Scheinkman decomposition, independently on the time horizon under consideration.

Marinacci and Severino (2018) formalize the eigenvalue-eigenvector problem of eq. (7) in a semimartingale framework by using weak time-derivatives on the time interval $[0, T]$ with the measure Q. In particular, they show that the risk-neutral price process given by $\pi_t(h_T)$ is the unique solution in $\mathcal{U}_0^1$ of the no-arbitrage pricing equation

$$\begin{cases} \mathcal{D} f_t = r\ f_t & t \in [0, T) \\ f_T = h_T \end{cases} \tag{8}$$

where $h_T \in L^1(\mathcal{F}_T, Q)$. In this problem the interest rate r is supposed to be constant.

Example. As an illustrative example we consider a Black and Scholes (1973) market with a riskless bond and a risky security whose prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t dW_t^P$$

under the physical measure. Here, $r, \mu \in \mathbb{R}$, $\sigma > 0$ and W_t^P is a standard Wiener process. It is apparent that the bond price satisfies problem (8). As for the risky asset, under the physical measure the Wiener process W_t^P exhibits null weak time-derivative because it is a martingale, but the drift of X_t differs from rX_t. However, moving to the risk-neutral measure Q, the dynamics of X_t becomes

$$dX_t = rX_t dt + \sigma X_t dW_t^Q, \tag{9}$$

where W_t^Q is a Q-Wiener process, and so $\mathcal{D}X = rX$. Hence, problem (8) is satisfied. In this arbitrage-free market both the risk-free and the risky security share the same drift, determined by the instantaneous rate.

The dynamics presented so far generalize to stochastic-rate settings by replacing r with the instantaneous rate Y_t. For example, Chapter 1 of Karatzas and Shreve (1998) shows how to obtain similar dynamics to eq. (9) in a semimartingale pricing framework with random rates. The same approach is exploited by Heath, Jarrow, and Morton (1992) in deriving no-arbitrage restrictions on the forward rate drift. In addition, Marinacci and Severino (2018) provide a generalization of problem (8) to stochastic rates in terms of weak time-derivatives assuming uniformly bounded instantaneous rates.

Nevertheless, there are two major flaws in the previous generalizations. First, the drift coefficient Y_t is unable to capture the stochastic discount factor growth rate over any time period, differently from the constant-rate case. Second, Y_t is a random process. Hence, it is not known ex ante (since it is floating over time) and so it forbids an eigenvalue-eigenvector formulation of the problem in the spirit of Hansen and Scheinkman (2009).

Our generalization of problem (8) solves these issues for any given dynamics of interest rates. We employ the forward measure instead of Q and T-bond yields in place of short-term rates to formulate a suitable eigenvalue-eigenvector problem. Moreover, we provide a conditional version of the differential problem defined on any time window $[s, T]$ by using the weak time-derivative in $[s, T]$ of Definition 1. The arising eigenvalue is an $\mathcal{F}_s$-measurable random variable (known at the beginning of the trading interval) and, consistently, the growth rate of the stochastic discount features the same $\mathcal{F}_s$-measurability.

To enter the details of our construction, we place ourselves in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, F^T)$, where we employ the forward measure. We solve in $\mathcal{U}_s^1$ the following pricing differential equation with random coefficient given by the yield to maturity $r_s^T \in L^0(\mathcal{F}_s)$, i.e.

$$\begin{cases} \mathcal{D}f_t = r_s^T \; f_t & t \in [s, T) \\ f_T = h_T \end{cases} \tag{10}$$

with $h_T \in L_s^1(\mathcal{F}_T, F^T)$. We refer to (10) as the *rate-adjusted pricing equation.*

Theorem 2 *There exists a unique solution of problem* (10) *in* $\mathcal{U}_s^1$*, given by*

$$\rho_t^T(s, h_T) = e^{-r_s^T(T-t)} \mathbb{E}_t^{F^T}[h_T] \qquad \forall t \in [s, T]. \tag{11}$$

Proof. See Appendix B. ■

We refer to $\rho_t^T(s, h_T)$ as the *rate-adjusted price* of h_T at time t in the interval $[s, T]$.

Theorem 2 is substantially an assessment of the conditional martingale property of the process $\{e^{r_s^T(T-t)} \rho_t^T\}_{t \in [s,T]}$ under F^T. This process is, in fact, the collection of forward prices for h_T in the period $[s, T]$.

At any instant $t \in [s, T]$, r_s^T is the only *average rate* employed by ρ_t^T for the valuation on h_T. The valuation instant is synchronous with the information set only for the initial ρ_s^T. In particular, ρ_s^T coincides with the no-arbitrage price π_s. The two coincide also at the terminal date.

In general, when $s < t < T$ the rate-adjusted price is different from the no-arbitrage price. Fixed any t, a bunch of valuations $\rho_t^T(s, h_T)$ are available, obtained by solving several problems as (10) defined on different time intervals $[s, T]$ with $s < t$. However, rate-adjusted prices with different starting points are consistent within them. Indeed, the martingale property of forward prices ensures that, for any $s_1 \leqslant s_2 \leqslant t$,

$$\mathbb{E}_{s_1}^{F^T}\left[e^{r_{s_2}^T(T-t)} \rho_t^T(s_2, h_T)\right] = \mathbb{E}_{s_1}^{F^T}\left[e^{r_{s_1}^T(T-t)} \rho_t^T(s_1, h_T)\right].$$

In addition, moving s to t from the left, if r_s^T converges in probability to r_t^T, then

$$\rho_t^T(s, h_T) \xrightarrow{P} \pi_t(h_T), \qquad s \to t^-.$$

This fact can be easily assessed by considering log prices and applying the continuous mapping theorem.

The proper link between ρ^T and π is given by

$$\rho_t^T(s, h_T) = e^{-\left(r_s^T - r_t^T\right)(T-t)} \pi_t(h_T) = e^{r_s^T(t-s)} \frac{\pi_s(1_T)}{\pi_t(1_T)} \pi_t(h_T). \tag{12}$$

This equality can also be read as a parity relation between ρ^T and π:

$$e^{r_s^T(T-t)} \rho_t^T(s, h_T) = e^{r_t^T(T-t)} \pi_t(h_T),$$

where both sides deliver the price at time t of a forward contract on h_T for date T.

The difference between rate-adjusted and no-arbitrage prices is genuinely due to the term structure of interest rates in the market: in case rates are constant, the distortion between ρ^T and π disappears. In fact, problem (10) generalizes the dynamics of no-arbitrage prices of problem (8) for constant rates of interest. ρ^T solves in $[s, T]$ the analogous differential equation of π, where interest rates are replaced by $\mathcal{F}_s$-measurable bond yields and the forward measure replaces the risk-neutral one. In this perspective rate-adjusted prices may be seen as a generalization of no-arbitrage prices in floating-rate markets. Moreover, when pure discount bonds are considered, $\rho_t^T(s, 1_T)$ is exactly the no-arbitrage price of the zero-coupon bond in a parallel market in which interest rates are fixed and equal to the yield over $[s, T]$.

Figure 1 considers U.S. Treasury bonds in January 1987 with increasing expiry T of $5, 10, 20$ and 30 years. We fix $s = 0$ at January 1987 and consider annual t up to the redemption date. Data are provided by the Federal Reserve Board at daily frequency: see Gürkaynak, Sack, and Wright (2007). In particular, the yield curve associated with U.S. Treasury bonds in January 1987 is increasing. In the four graphs of Figure 1 we plot the rate-adjusted prices $\rho_t^T(0, 1_T)$ of these securities by using dashed lines. Moreover, we use solid lines for the ex-post realizations of no-arbitrage prices $\pi_t(1_T)$ observed in later years in the market. Treasury bonds rate-adjusted and no-arbitrage prices are indistinguishable at the initial date and near maturity. Different values of r_0^T may, however, induce an overestimation or underestimation of bond prices, which can be evaluated in future periods according to the actual realizations of bond prices in the market. Specifically, $\rho_t^T(0, 1_T) < \pi_t(1_T)$ if and only if $r_0^T > r_t^T$, a property that holds for any nonnegative payoff h_T, too. The relation between yields at different times determines, indeed, the discrepancy between rate-adjusted and no-arbitrage valuation.

We illustrate additional comparisons between rate-adjusted and no-arbitrage prices in Section 5, where we consider, for instance, European options on bonds.

3.2 Rate-adjusted prices as indifference prices

Rate-adjusted prices can be interpreted as *indifference prices* that allow investors to hedge from interest rates variability. We bolster this intuition by describing a simple self-financing strategy in our market.

We consider an investor who incurs an expenditure of $\pi_s(1_T)$ at time s in order to buy a self-financing portfolio that delivers some units of a derivative with payoff h_T at maturity. Specifically, at date s she buys a zero-coupon bond with expiration T. At a later time t she rebalances her allocation, worth $\pi_t(1_T)$, by purchasing the amount $\pi_t(1_T)/\pi_t(h_T)$ of derivative, which is traded in the market at the no-arbitrage price. Then, she holds this portfolio until maturity.

We suppose that another investor faces the same market. She is less sophisticated than the previous one and pretends that the zero-coupon bond is traded at the fixed rate r_s^T. Her belief is consistent with the observation of the T-bond price at time s, that is

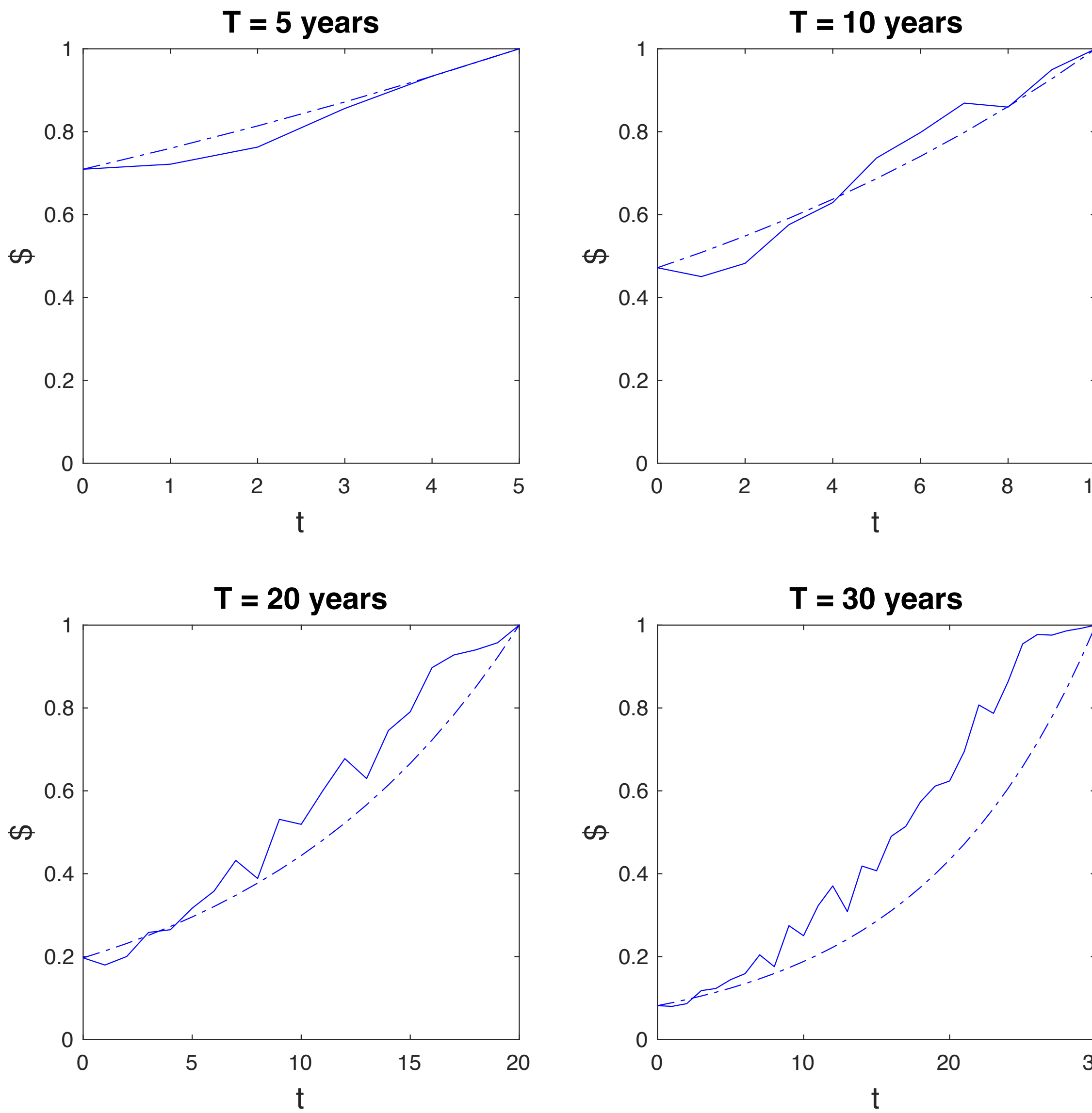


Figure 1: We fix $s = 0$ at January 1987 and we consider increasing annual maturities T of U.S. Treasury bonds. Dashed lines depict the values of $\rho_t(0, 1_T)$ for any year t in $[0, T]$. Accordingly, solid lines represent the (ex-post) realizations of no-arbitrage prices $\pi_t(1_T)$ at any year t until expiration.

$\pi_s(1_T) = e^{-r_s^T(T-s)}$, which coincides with $\rho_s^T(s, 1_T)$. Similarly to the first agent, she plans to go long on the pure discount bond at time s and to entirely liquidate her position at t in order to purchase all the units of derivative that she can afford.

Since the second investor disregards the term structure of T-bond prices, she expects a position worth $e^{-r_s^T(T-t)}$ at time t and, consequently, an erroneous amount of derivative at maturity. However, the two agents would end up with the same number of units of h_T if the second investor used ρ^T for pricing the derivative. Indeed, $\rho_t^T(s, h_T)$ is the theoretical price of h_T that makes the terminal values of both portfolios coincide.

This description substantiates the hedging nature of ρ^T. Although the first investor properly exploits the variability of rates, the second one pretends to face a flat term structure. Given an identical initial expenditure, they obtain the same outcome if the less sophisticated agent employs rate-adjusted prices for the valuation of marketed payoffs.

3.3 Rate-adjusted prices as conversion prices

We now provide an interpretation of rate-adjusted prices in terms of conversion prices of mandatory convertible bonds with contingent conversion prices. To be in line with the previous notation, we assume that the payoff h_T represents a stock price and we denote it by X_T.

We consider a convertible bond issued at time s with maturity T, unitary price and fixed rate r_s^T. At a given time $t < T$, the security is compulsorily converted into a proper amount of shares. The conversion price and ratio are determined at time t by no-arbitrage considerations.

Suppose that at time s the issuer of the hybrid security purchases $1/\pi_s(1_T)$ units of a pure discount bond (traded in the market) with redemption date T. The total cashflow at s is null. Immediately before the conversion, the value of the position is $\pi_t(1_T)/\pi_s(1_T)$ while the convertible bond is worth $e^{r_s^T(t-s)}$. On the one hand, the holdings of the issuer at time t correspond to $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$ shares with price $\pi_t(X_T)$. On the other, the bond conversion takes place according to the relation $e^{r_s^T(t-s)} = p_t q_t$, where p_t and q_t are $\mathcal{F}_t$-measurable conversion price and ratio. The absence of arbitrage opportunities implies that q_t needs to coincide with the amount of shares $\pi_t(1_T)/(\pi_s(1_T)\pi_t(X_T))$. Otherwise, an initial null expenditure would ensure a positive outcome (in terms of stocks) at terminal date T. Since $p_t = e^{r_s^T(t-s)}/q_t$, we deduce that p_t takes the expression of eq. (12). As a result, p_t equals the rate-adjusted price $\rho_t^T(s, X_T)$.

The construction of the previous hybrid asset characterizes ρ^T as a conversion price from bonds to stocks in an arbitrage-free market. The reasoning highlights the ability of ρ^T to quantify the risk exposure when financing through fixed-rate securities in a market with stochastic rates where zero-coupon bonds are traded.

3.4 Role of the yield r_s^T in the rate-adjusted problem

Problem (10) is not the sole way to generalize the differential relation $\mathcal{D}\pi = r\pi$ of problem (8). Indeed, under the forward measure, any process $f \in \mathcal{U}_s^1$ defined by $f_t = e^{-\xi_s(T-t)}\mathbb{E}_t^{F^T}[h_T]$ with $\xi_s \in L^0(\mathcal{F}_s)$ solves

$$\begin{cases} \mathcal{D}f_t = \xi_s \ f_t & t \in [s, T) \\ f_T = h_T \end{cases} \tag{13}$$

where $h_T \in L_s^1(\mathcal{F}_T, F^T)$.

For instance, assume that the pure discount bond price $\pi_s(1_T)$ is differentiable with respect to maturity T. We can set ξ_s as the instantaneous forward rate with settlement date T contracted at time s, i.e.

$$\xi_s = -\frac{\partial \log \pi_s\left(1_T\right)}{\partial T} = r_s^T + \frac{\partial r_s^T}{\partial T}(T - s).$$

The associated process f satisfies the parity relation with the no-arbitrage price π, namely $e^{\xi_s(T-t)}f_t = e^{r_t^T(T-t)}\pi_t(h_T)$, and both terms of the equality match the forward price of h_T at t. In addition, in case interest rates are constant, ξ_s collapses to the instantaneous rate and f specializes to the no-arbitrage price. However, although conceptually attracting, the instantaneous forward rate has an important drawback: at the initial date s, f_s is different from the no-arbitrage price π_s when rates are time-varying.

Another interesting parametrization is provided by $\xi_s = \mathbb{E}_s[Y_T]$. This quantity actually coincides with the instantaneous forward rate if the expectation hypothesis of Fisher (1930) holds. The solution f of problem (13) can be easily determined. As in the previous example, ξ_s reduces to the instantaneous rate in case rates are constant and, accordingly, f becomes indistinguishable from π. Nevertheless, when interest rates are floating over time, there is an initial mispricing because f_s is different from π_s.

The initial f_s equals the no-arbitrage price π_s if and only if $\xi_s = r_s^T$. This fact motivates the choice of the yield to maturity r_s^T in the formalization of problem (10) instead of alternative generalizations. Indeed, the arising solution ρ^T exactly provides the no-arbitrage price at instant s. Furthermore, ρ^T is the unique process – among all possible f that solve problems as (13) – that features this correctness property.

3.5 Long-term relations between rate-adjusted and no-arbitrage prices

We now investigate the relation between ρ_t^T and π_t when maturity T increases. Although yields to maturity r_s^T and r_t^T converge to the same long-term yield, the asymptotic relations between ρ_t^T and π_t are rather delicate. As discussed in Subsection 2.2, we adopt the assumptions of Qin and Linetsky (2017).

Proposition 3 *For any $s > 0$ and $t > s$*

$$\frac{\rho_t^T(s, h_T) - \pi_t(h_T)}{\mathbb{E}_t^{F^T}[h_T]} \xrightarrow{P} 0, \qquad T \to +\infty$$

and

$$\frac{\log \rho_t^T(s, h_T) - \log \pi_t(h_T)}{T - t} \xrightarrow{P} 0, \qquad T \to +\infty.$$

Proof. See Appendix B. ■

The asymptotic behaviours of Proposition 3 are valid for any choice of s and $t > s$, which are fixed before taking the limit across increasing horizons. However, convergences are taken just after a proper rescaling of rate-adjusted and no-arbitrage prices. The normalization of the first asymptotic result is actually irrelevant when pure discount bonds are under scrutiny. The second result employs the tenor to rescale the difference between ρ^T and π in logs.

We now concentrate on the ratio between rate-adjusted and no-arbitrage prices.

Proposition 4 *For all $s > 0$ and $t > s$*

$$\mathbb{E}_s^{F^T}\left[\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)}\right] \xrightarrow{P} e^{\left(r^\infty - r_s^t\right)(t-s)}, \qquad T \to +\infty$$

and

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty.$$

Proof. See Appendix B. ■

The difference between the long-term yield and the bond yield r_s^t determines the limit for the expected ratio between ρ_t^T and π_t under the forward measure. In addition, this ratio turns out to have a well-defined long-run limit in probability. Asymptotically ρ_t^T differs from π_t by a multiplicative $\mathcal{F}_t$-measurable factor associated with the discounted long bond.

3.6 Rate-adjusted prices as long-term prices

As described by eq. (4) in Subsection 2.3, the long-term growth rate of the pricing kernel $M_{s,t}$ is the long-term yield r^∞. Since r^∞ is the limit of r_s^T when T goes to infinity and r_s^T is the leading parameter of problem (10), we move T to infinity in this problem and we analyze the behaviour of the solutions over increasing horizons. We start from solving the *long-term* rate-adjusted problem

$$\begin{cases} \mathcal{D}f_t = r^{\infty}\ f_t & t \in [s,T) \\ f_T = h_T \end{cases} \tag{14}$$

with $h_T \in L^1_s(\mathcal{F}_T, F^{\infty})$. Differently from problem (10), here the long-term yield replaces r^T_s and the long-term forward measure is employed.

Theorem 5 *There exists a unique solution of problem* (14) *in* $\mathcal{U}^1_s$*, given by*

$$\rho^{\infty}_t(s,h_T) = e^{-r^{\infty}(T-t)}\mathbb{E}^{F^{\infty}}_t[h_T] \qquad \forall t \in [s,T]. \tag{15}$$

Proof. See Appendix B. ■

We refer to ρ^{∞}_t as the *long-term rate-adjusted price* of h_T at time t in the interval $[s,T]$.

Problem (14) and problem (10) differ in two main aspects. The first one is the horizon of the forward measure, which is either infinite or finite. The second one is the randomness of the multiplier. Indeed, r^{∞} is a number while r^T_s is properly an $\mathcal{F}_s$-measurable random variable.

After solving problems (10) and (14) separately, we address the issue of the convergence of ρ^T to ρ^{∞} when the horizon increases. In particular, we solve a sequence of differential problems related to a term structure of horizons, whose solutions tend to the solution of the long-term rate-adjusted problem.

In order to do so, we disentangle the instant in which the payoff under scrutiny is paid from the horizon. We assume that the maturity of the security under consideration is τ, while the horizon is $T \geqslant \tau$. The rate-adjusted pricing problem of Subsection 3.1 can, then, be rewritten as

$$\begin{cases} \mathcal{D}f_t = r^T_s\ f_t & t \in [s,\tau) \\ f_\tau = h_\tau \end{cases} \tag{16}$$

with $h_\tau \in L^1_s(\mathcal{F}_\tau, F^T)$. If $\tau = T$, we precisely retrieve problem (10). In the more general formulation considered here, the unique solution in $\mathcal{U}^1_s$ over $[s,\tau)$ is

$$\rho^T_t(s,h_\tau) = e^{-r^T_s(\tau-t)}\mathbb{E}^{F^T}_t[h_\tau]$$

for all $t \in [s,\tau]$. Finally, we investigate the convergence of $\rho^T_t(s,h_\tau)$ when T increases. Under mild assumptions, the solution of problem (16) converges in probability to the long-term rate-adjusted price that solves problem (14).

Proposition 6 *Suppose that* $h_\tau \in L^1_s(\mathcal{F}_\tau, F^T)$ *for all* $T \geqslant \tau$ *and* $G^T_{t,\tau}h_\tau$ *is convergent in* L^1 *when* T *goes to infinity for all* $t \in [s,\tau]$*. Then, for all* $t \in [s,\tau]$,

$$\rho^T_t(s,h_\tau) \xrightarrow{P} \rho^{\infty}_t(s,h_\tau), \qquad T \to +\infty.$$

Proof. See Appendix B. ■

As a result, ρ^∞ can be properly interpreted as a rate-adjusted price for the long run. Indeed, the asset valuation through ρ^∞ exploits r^∞, which constitutes the stochastic discount factor long-term growth rate.

4 Pricing kernel growth

From the definition of problem (10) it is clear that $\mathcal{D}\rho^T$ is weakly time-differentiable in $[s, T]$. The same also holds for weak time-derivatives of higher orders. As a result, the rate-adjusted price ρ^T is infinitely weakly time-differentiable and so it belongs to $\mathcal{U}_s^\infty$. A parallel reasoning ensures the infinite weak differentiability of ρ^∞.

Moreover, by eq. (6), the weak time-derivative in $[s, T]$ defines an L^0-linear operator $\mathcal{D} : \mathcal{U}_s^\infty \to \mathcal{U}_s^\infty$ and ρ^T satisfies the eigenvalue-eigenvector problem

$$\mathcal{D}\rho^T = r_s^T \ \rho^T, \tag{17}$$

where the eigenvalue belongs to $L^0(\mathcal{F}_s)$. Accordingly, ρ^∞ solves

$$\mathcal{D}\rho^\infty = r^\infty \ \rho^\infty \tag{18}$$

where the eigenvalue is a positive number.

As in Hansen and Scheinkman (2009), we suppose that the payoff h_T is positive so that ρ^T and ρ^∞ are positive. Hence, ρ^T and ρ^∞ are *principal eigenvectors* related to r_s^T and r^∞ respectively. This property is key to the economic theory because it is in line with the Perron-Frobenius theory, usually employed in Markovian environments and successfully applied by Ross (2015) and Qin and Linetsky (2016). Indeed, when rates are constant, the principal eigenvalue associated with a differential price operator turns out to be the growth rate of the stochastic discount factor (see Hansen and Scheinkman, 2009).

In this section we analyze the relation between the eigenvalues of several differential pricing problems and pricing kernel growth rates in our stochastic-rate market. To build our theory, we introduce the notion of rate-adjusted pricing kernel in Subsection 4.2.

4.1 Finite- and infinite-horizon pricing kernel decomposition

The long-term eigenvalue problem of eq. (18) actually exploits the pricing kernel long-term growth rate, that is the long-term yield. Indeed, as already shown by eq. (4), $M_{s,t}$ satisfies the long-run decomposition

$$M_{s,t} = e^{-r^\infty (t-s)} \ \frac{b_s^\infty}{b_t^\infty} \ G_{s,t}^\infty.$$

The question, here, is whether an analogous property holds for the T-horizon problem of eq. (17). We need first to characterize the finite-horizon pricing kernel growth rate.

As introduced in Subsection 2.3, the pricing kernel $M_{s,t}$ satisfies

$$M_{s,t} = \frac{e^{-r_s^T(T-s)}}{e^{-r_t^T(T-t)}} G_{s,t}^T = \frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} \frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} G_{s,t}^T$$

for all $T > t + s$. In the next proposition we assess the asymptotic behaviours of the last three factors. Note, in particular, that the ratio $\pi_s(1_T)/\pi_s(1_{T-t})$ is the return on a forward rate agreement between $T - t$ and T contracted at date s.

Proposition 7 *For all $s > 0$ and $t > s$*

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} \xrightarrow{P} e^{-r^\infty(t-s)}, \qquad T \to +\infty$$

$$\frac{e^{-r_s^T s}\pi_s(1_{T-t})}{\pi_t(1_T)} \xrightarrow{P} \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty$$

$$G_{s,t}^T \xrightarrow{P} G_{s,t}^\infty, \qquad T \to +\infty.$$

Proof. See Appendix C. ■

The first convergence of Proposition 7 is interesting for many aspects. In addition to being a limit of bond yields, r^∞ is also a limit of continuously compounded forward rates contracted at s, as the convergence of $\pi_s(1_T)/\pi_s(1_{T-t})$ suggests. This approach is reminiscent of the construction of Backus, Gregory, and Zin (1989) and of the methodology of Alvarez and Jermann (2005), who highlight the informativeness of prices of bonds with distant maturities on pricing kernel persistence. The term $e^{-r^\infty(t-s)}$ determines, in fact, the growth of the pricing kernel in the time interval $[s, t]$ embedded in an arbitrarily large time horizon.

The second convergence of Proposition 7 involves the transient component of $M_{s,t}$. The limit of this factor is captured by the ratio of the discounted values of the long bond.

Finally, the third convergence regards the permanent (or martingale) component of the pricing kernel. At any finite horizon, this term consists of the Radon-Nikodym density of the T-forward measure. When the horizon is infinite, the long-term forward measure appears.

Consistently with the three simultaneous convergences of Proposition 7, when the horizon is finite we call growth term of $M_{s,t}$ the first ratio of the proposition, i.e.

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{-r_s^T(T-2s)+r_s^{T-t}(T-t-s)}. \tag{19}$$

Hence, $M_{s,t}$ features a random growth term in $L^0(\mathcal{F}_s)$ determined by bond yields computed at time s. In case interest rates are constant, this quantity actually reduces to $e^{-r(t-s)}$, which captures the growth of the pricing kernel $e^{-r(t-s)}L_{s,t}$.

As it is apparent from eq. (19), the growth rate of $M_{s,t}$ is not exactly r_s^T and so, differently from the long-term yield, it does not agree with the eigenvalue problem $\mathcal{D}\rho^T = r_s^T \rho^T$. Therefore, in order to isolate the growth rate r_s^T in finite horizons, we introduce the notion of *rate-adjusted pricing kernel*, which parallels the one of rate-adjusted prices.

4.2 Rate-adjusted pricing kernel

Given any payoff h_T, we can write no-arbitrage and rate-adjusted prices at time t as

$$\pi_\tau(h_T) = \mathbb{E}_\tau[M_{\tau,T}h_T], \qquad \rho_\tau^T(s,h_T) = \mathbb{E}_\tau\left[N_{\tau,T}^T h_T\right],$$

where, for all τ, t in $[s,T]$ with $\tau \leqslant t$,

$$M_{\tau,t} = \frac{e^{r_t^T(T-t)}}{e^{r_\tau^T(T-\tau)}} G_{\tau,t}^T, \qquad N_{\tau,t}^T = \frac{e^{r_s^T(T-t)}}{e^{r_s^T(T-\tau)}} G_{\tau,t}^T.$$

We call $N_{s,t}^T$ the *rate-adjusted pricing kernel.* In particular,

$$N_{s,t}^T = e^{\left(r_s^T - r_t^T\right)(T-t)} M_{s,t}$$

and it coincides with $M_{s,t}$ when rates are constant. The adjustment coefficient between $N_{s,t}^T$ and $M_{s,t}$ is the inverse of the one between π and ρ^T pointed out in eq. (12). In addition, the expected values of $M_{s,t}$ and $N_{s,t}^T$ differ only in the yields r_s^t and r_s^T:

$$\mathbb{E}_s[M_{s,t}] = e^{-r_s^t(t-s)}, \qquad \mathbb{E}_s\left[N_{s,t}^T\right] = e^{-r_s^T(t-s)}.$$

Since $N_{s,t}^T = e^{-r_s^T(t-s)}G_{s,t}^T$, its growth rate is r_s^T, in agreement with the rate-adjusted eigenvalue problem $\mathcal{D}\rho^T = r_s^T\rho^T$. Moreover, differently from $M_{s,t}$, the rate-adjusted pricing kernel is explicitly dependent on the horizon T under scrutiny. In the next proposition we establish the convergence of $N_{s,t}^T$ when T becomes arbitrarily large.

Proposition 8 *For all $s > 0$ and $t > s$,*

$$N_{s,t}^T \xrightarrow{P} e^{-r^\infty(t-s)} G_{s,t}^\infty, \qquad T \to +\infty.$$

Proof. See Appendix C. ■

Therefore, we define the *long-term rate-adjusted pricing kernel* by

$$N_{s,t}^\infty = e^{-r^\infty(t-s)} G_{s,t}^\infty \tag{20}$$

and r^{∞} naturally arises as growth rate for $N_{s,t}^{\infty}$.

A formalization of the fact that r_s^T and r^{∞} are the rate-adjusted pricing kernel growth rates for finite (or infinite) horizons can be easily obtained in differential terms. Indeed, when the horizon is finite, we can consider the problem

$$\begin{cases} \mathcal{D}f_t = -r_s^T f_t & t \in [s,T] \\ f_s = 1 \end{cases} \tag{21}$$

under the physical measure. When the horizon is infinite, we analyze

$$\begin{cases} \mathcal{D}f_t = -r^{\infty}\ f_t & t \in [s,+\infty) \\ f_s = 1 \end{cases} \tag{22}$$

still under P. The last problem uses weak time-derivatives in $[s,+\infty)$ that can be defined by replacing T with $+\infty$ in Definition 1. The module $\mathcal{U}_s$ modifies accordingly by omitting the left-continuity of processes u at T but additionally requiring that $\int_s^{+\infty} \mathbb{E}_s[|u_\tau|]d\tau$ belongs to $L^0(\mathcal{F}_s)$.

Theorem 9 *Under the measure P, $\{N_{s,t}^T\}_t$ solves problem* (21) *in $\mathcal{U}_s^1$ with finite T and $\{N_{s,t}^{\infty}\}_t$ solves problem* (22) *in $\mathcal{U}_s^1$ with infinite T.*

Proof. See Appendix C. ■

The last theorem formalizes the growth terms r_s^T and r^{∞} for the rate-adjusted pricing kernel at finite and infinite horizons. The results are consistent with the eigenvalue problems $\mathcal{D}\rho^T = r_s^T \rho^T$ and $\mathcal{D}\rho^{\infty} = r^{\infty}\rho^{\infty}$ satisfied by rate-adjusted prices.

Problems (21) and (22) generalize the differential relation $\mathcal{D}M = -rM$ which is satisfied by the pricing kernel when interest rates are constantly equal to r. When rates are stochastic, in standard diffusive models, the pricing kernel follows the dynamics

$$dM_{s,t} = -Y_t M_{s,t}dt - \nu_t M_{s,t} dW_t^P,$$

where ν_t is the market price of risk (see, for instance, Subsection 5.2.2). The stochastic rate is present in the drift of the pricing kernel. However, any Y_t alone is not able to capture the growth rate of $M_{s,t}$ and it is not known ex ante. This last feature forbids any eigenvalue-eigenvector formulation of the problem of identifying a growth rate for $M_{s,t}$.

Problems (21) and (22) are not satisfied by the pricing kernel $M_{s,t}$ when rates of interest are stochastic as well as no-arbitrage prices do not solve problems (10) and (14). The deep reason for these failures can be understood through the lenses of the Hansen-Scheinkman decomposition.

By comparing eq. (20) with the long-term decomposition of the pricing kernel in eq. (4), it is apparent that $N^{\infty}_{s,t}$ features the same growth rate r^{∞} of $M_{s,t}$, as well as the same martingale component $G^{\infty}_{s,t}$. Nevertheless, the transitory component of $N^{\infty}_{s,t}$ is deterministic and equal to 1. Therefore, employing rate-adjusted prices for asset valuation means using a stochastic discount factor that is free from transitory effects in its long-term Hansen-Scheinkman decomposition. See also the previous eq. (5) that highlights the role of the transient component of $M_{s,t}$ in the price of any marketed payoff.

When interest rates are constant, the transient component of the stochastic discount factor is 1 and so the difference between $M_{s,t}$ and $N^{\infty}_{s,t}$ vanishes. When rates are stochastic, a trivial temporary term can be retrieved only in $N^{\infty}_{s,t}$. For this reason, rate-adjusted prices allow us to generalize both price and stochastic discount factor dynamics from the constant-rate case. Rate-adjusted prices aggregate infinitesimal randomness to long-run risk exposure because $N^{T}_{s,t}$ is free from any transitory component. Moreover, bond yields are the proper financial variables that translate local riskiness to long-run risks through rate-adjusted prices.

We summarize our main findings in Table 1.

	Constant rate	**Stochastic rates**	
		finite T	**infinite T**
Numéraire	money market	zero-coupon bond	long bond
Measure	Q	F^T	F^{∞}
Yield	r	r^T_s	r^{∞}
Price	π	ρ^T	ρ^{∞}
Return-rate relation	$\mathcal{D}\pi = r\pi$	$\mathcal{D}\rho^T = r^T_s \rho^T$	$\mathcal{D}\rho^{\infty} = r^{\infty}\rho^{\infty}$
Pricing kernel growth rate	r	r^T_s	r^{∞}

Table 1: Comparison of numéraires, pricing relations and pricing kernel growth rates between constant and stochastic rates. Finite and infinite horizons are considered. In the stochastic case rate-adjusted prices and stochastic discount factors are considered.

5 Affine interest rate models

We provide an illustration of our theory in arbitrage-free markets with diffusive short-term rates. After specializing to affine term structure models, we illustrate the differences between rate-adjusted and no-arbitrage prices of pure discount bonds and bond European call options. We also devote some attention to the Feynman-Kaç partial differential equations satisfied by the two prices. We finally highlight the dynamics of the pricing kernels (risk-adjusted or not) and their growth terms. Subsection 5.1 considers a fixed-income market while Subsection 5.2 involves a market with both stocks and zero-coupon bonds.

5.1 Pricing in a fixed-income market

We start comparing the drifts of zero-coupon bond prices under the measures P, Q and F^T in a fixed-income market. Then, we make a comparison with rate-adjusted prices. Indeed, when the processes under scrutiny are weakly time-differentiable, the drift is the weak time-derivative, as shown in Marinacci and Severino (2018).

We assume that instantaneous rates follow the diffusion process $dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t^P$ in the time window $[s, T]$. Here, μ and σ are measurable functions of t and Y_t, and W_t^P denotes a Wiener process under the physical measure. The pure discount T-bond price at time t is function of t and Y_t. Therefore, Itô's formula permits to determine drift and diffusion coefficient of $\pi_t(1_T)$, so that we can write

$$\frac{d\pi_t\left(1_T\right)}{\pi_t\left(1_T\right)} = \tilde{\mu}\left(t, Y_t\right)dt + \tilde{\sigma}\left(t, Y_t\right)dW_t^P.$$

We indicate by ν the market price of risk process $\nu_t = (\tilde{\mu}(t, Y_t) - Y_t)/\tilde{\sigma}(t, Y_t)$. By Girsanov's theorem, we build a Wiener process under the risk-neutral measure Q through the stochastic differential $dW_t^Q = dW_t^P + \nu_t dt$. The measure Q triggers a drift change in the evolution of bond prices:

$$\frac{d\pi_t\left(1_T\right)}{\pi_t\left(1_T\right)} = Y_t\ dt + \tilde{\sigma}\left(t, Y_t\right)dW_t^Q.$$

Under Q the drift coefficient of no-arbitrage bond prices coincides with the instantaneous rate. When changing the numéraire to the pure discount T-bond, the dynamics of $\pi_t(1_T)$ under the forward measure become

$$\frac{d\pi_t\left(1_T\right)}{\pi_t\left(1_T\right)} = \left(Y_t + \tilde{\sigma}^2\left(t, Y_t\right)\right)dt + \tilde{\sigma}\left(t, Y_t\right)dW_t^{F^T}, \tag{23}$$

where $W_t^{F^T}$ is a Wiener process under F^T satisfying $dW_t^{F^T} = dW_t^Q - \tilde{\sigma}(t, Y_t)dt$. See Chapter 3 of Brigo and Mercurio (2006). In eq. (23) the drift change is apparent. Differently from the Q-dynamics, the drift under F^T depends on the diffusion coefficient of the bond price itself. This feature is not present in T-bond rate-adjusted prices because their differential satisfies

$$\frac{d\rho_t^T\left(s, 1_T\right)}{\rho_t^T\left(s, 1_T\right)} = r_s^T\ dt \tag{24}$$

for any measure under consideration. In agreement with Theorem 2, the weak time-derivative in $[s,T]$ of $\rho_t^T(s,1_T)$ is $r_s^T \rho_t^T(s,1_T)$, while under F^T the candidate weak time-derivative of $\pi_t(1_T)$ is $(Y_t+\tilde{\sigma}^2(t,Y_t))\pi_t(1_T)$. Since $Y_t+\tilde{\sigma}^2(t,Y_t)$ is floating over time, under F^T it is unlikely to find an eigenvalue-eigenvector formulation as problem (10) by using no-arbitrage prices when rates of interest are stochastic.

5.1.1 Zero-coupon bonds in Vasicek (1977)

In one-factor affine term structure models – as Vasicek (1977), Cox, Ingersoll, and Ross (1985), Ho and Lee (1986) or Hull and White (1990) – the price of a zero-coupon bond with expiration T depends on the instantaneous rate Y_t through the exponential relation $\pi_t(1_T) = e^{A(t,T)-B(t,T)Y_t}$, where A and B are deterministic functions. Accordingly, the yield to maturity is affine in Y_t. By Itô's differential rule, the diffusion coefficient of the no-arbitrage price $\pi_t(1_T)$ is $\tilde{\sigma}(t,Y_t) = -B(t,T)\sigma(t,Y_t)$ and so, under the forward measure, its drift depends on the volatility of the underlying instantaneous rate:

$$\frac{d\pi_t\left(1_T\right)}{\pi_t\left(1_T\right)} = \left(Y_t + B^2(t,T)\sigma^2\left(t,Y_t\right)\right)dt - B(t,T)\sigma\left(t,Y_t\right)dW_t^{F^T}.$$

However, the dynamics of the related rate-adjusted price is still the one of eq. (24).

As an example, we consider the Vasicek (1977) model, in which the coefficients of the instantaneous rate are $\mu(t,Y_t) = k\theta - (k-\sigma\xi)Y_t$ and $\sigma(t,Y_t)=\sigma$, where $k,\theta,\sigma,\xi > 0$ and the market price of risk is ξY_t. Thus, the evolution of Y_t under Q is given by

$$dY_t = k\left(\theta - Y_t\right)dt + \sigma dW_t^Q. \tag{25}$$

The short-term rate is mean-reverting towards the value θ in the long run at a speed dictated by k. In addition, volatility is constant over time. Under this specification,

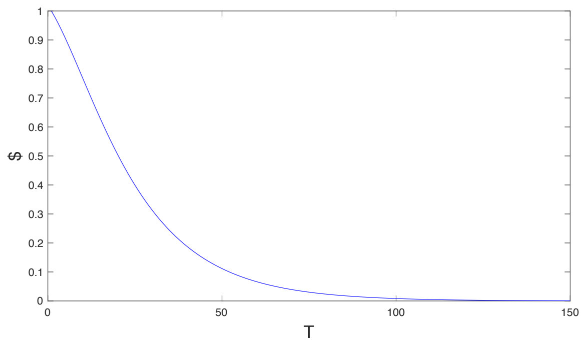
$$A(t,T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(B(t,T) - T + t\right) - \frac{\sigma^2}{4k}B^2(t,T)$$

and $B(t,T) = (1-e^{-k(T-t)})/k$, as described in Section 3.2 of Brigo and Mercurio (2006). Therefore, the drift coefficient of the no-arbitrage bond price under F^T is

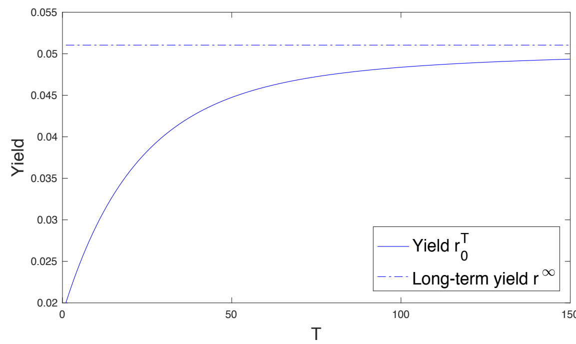
$$Y_t + \frac{\sigma^2}{k^2}\left(1 - e^{-k(T-t)}\right)^2.$$

When the horizon T becomes infinitely large, this term converges a.s. to $Y_t+\sigma^2/k^2$. On the contrary, the drift parameter r_s^T of the rate-adjusted price converges a.s. to the long-term yield $r^\infty = \theta - \sigma^2/2k^2$.

Figure 2 displays the term structure of no-arbitrage prices and yields of a pure discount bond in a simulated Vasicek model, computed at time zero. For the numerical exercise we



(a) Zero-coupon bond prices term structure.



(b) Yield-to-maturity term structure.

Figure 2: The left and right panel depict no-arbitrage pure discount bond prices (at time zero) and yields to maturity with respect to increasing horizons T under Vasicek specifications. The dashed horizontal line represents the long-term yield r^∞.

fix the parameters $k = 0.0331$, $\theta = 0.0967$, $\sigma = 0.01$ as estimated by Nowman (1997) in Table III with monthly data and we set $\xi = 0.2$ and $Y_0 = 0.02$. When horizon T gets larger and larger we visualize the convergence of yields to maturity to the long-term yield r^∞. On the contrary, left panel of Figure 3 shows prices $\pi_t(1_T)$ and $\rho_t^T(0, 1_T)$ of pure discount T-bonds when the horizon is fixed. The rate-adjusted price overlaps the no-arbitrage price both in the short-term and in the proximity of the redemption date, consistently with the discussion in Subsections 3.1 and 3.4. A departure from $\pi_t(1_T)$ is sizeable in the middle of the trading window. Moreover, due to its simple expression, the rate-adjusted price turns out to be generally smoother than the risk-neutral price. In Appendix D we repeat the exercise by simulating a Cox, Ingersoll, and Ross (1985) model.

5.1.2 Call options on zero-coupon bonds in Vasicek (1977)

So far we discussed no-arbitrage prices of zero-coupon bonds under Vasicek assumptions. However, also derivative pricing has been extensively studied in this setting by using forward measures. A germane example is provided by Jamshidian (1989), which extends Black and Scholes (1973) approach to options on pure discount bonds.

Thus, consider a European call option with expiration τ and strike price c over a zero-coupon bond with maturity $T > \tau$. When short-term rates follow Vasicek dynamics, the no-arbitrage price of the option at any time t between 0 and τ is

$$\pi_t(h_\tau) = \pi_t(1_T)\mathcal{N}(q) - c\pi_t(1_\tau)\mathcal{N}(q - \hat{\sigma}),$$

where $\mathcal{N}$ denotes the cumulative distribution function of a standard Gaussian,

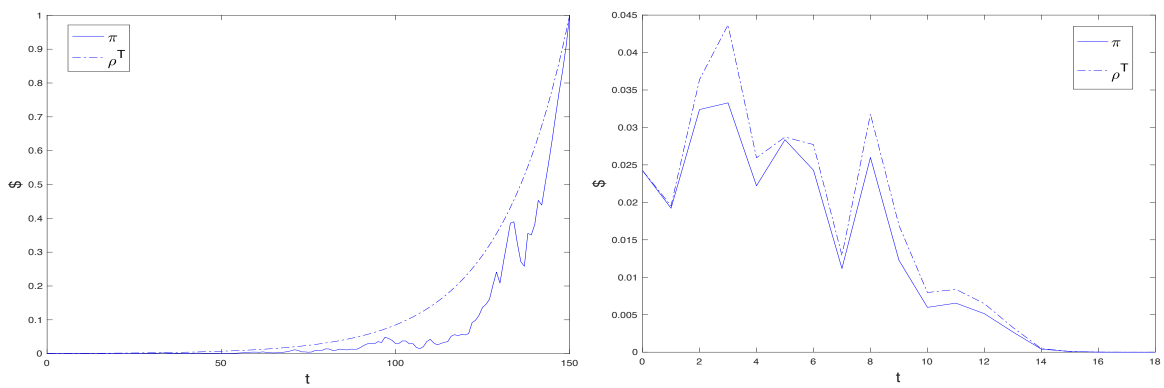
$$q = \frac{\log(\pi_t(1_T)) - \log(c\pi_t(1_\tau)) + \hat{\sigma}^2/2}{\hat{\sigma}},$$

$$\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2(k-\sigma\xi)(\tau-t)}}{2(k-\sigma\xi)} \frac{\left(1 - e^{-(k-\sigma\xi)(T-\tau)}\right)^2}{(k-\sigma\xi)^2}.$$

See the derivation in Jamshidian (1989).

As illustrated in Subsection 3.1, the rate-adjusted price of the option can be easily obtained from the no-arbitrage price through the relation $\rho_t^\tau(0, h_\tau) = e^{-(r_0^\tau - r_t^\tau)(\tau-t)}\pi_t(h_\tau)$.

We plot both price processes in the right panel of Figure 3, where we set $T = 36$ months, $\tau = 18$ months and $c = 0.5$. The two quantities almost coincide in the first months and near the option expiration τ. As expected, the difference between the two may be appreciated just during intermediate months.



(a) Zero-coupon bond. (b) Call option on a zero-coupon bond.

Figure 3: Left panel: values of $\rho_t^T(0, 1_T)$ and of a realization of $\pi_t(1_T)$ in Vasicek. Right panel: realizations of $\rho_t^\tau(0, h_\tau)$ and $\pi_t(h_\tau)$ for a European call option on a pure discount bond as described in the text. No-arbitrage prices are represented by solid lines and rate-adjusted prices by dashed lines.

5.1.3 Long-term relations for any payoff

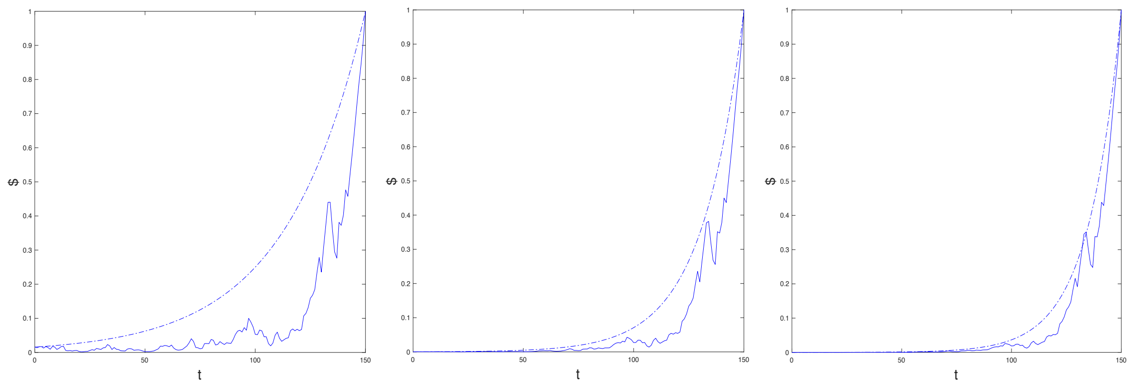
We now consider a generic attainable payoff h_T in a market with exponential affine interest rates. The ratio between the rate-adjusted price and the no-arbitrage price depends only on the instantaneous rates at instants s and t:

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} = e^{\{A(s,T) - B(s,T)Y_s\}\frac{T-t}{T-s} - A(t,T) + B(t,T)Y_t}.$$

In addition, the long-run relation between ρ^T and π of Proposition 4 can be determined explicitly under Vasicek assumptions:

$$\frac{\rho_t^T(s, h_T)}{\pi_t(h_T)} \xrightarrow{a.s.} e^{-\frac{1}{k}(Y_s - Y_t)}, \qquad T \to +\infty.$$

This limit is determined by the speed parameter k. If k is high, the two prices are almost indistinguishable for large maturities.



(a) $k = 0.025$. (b) $k = 0.035$. (c) $k = 0.045$.

Figure 4: Values of $\rho_t^T(0, 1_T)$ and of a realization of the no-arbitrage price $\pi_t(1_T)$ of a zero-coupon bond with fixed maturity T for any t in $[0, T]$. In all panels, the two prices are represented by a dashed and a solid line, respectively. The employed Vasicek parameters are the ones described in the text except for k that takes increasing values.

More is actually true: for any (finite) horizon T, the ratio between ρ_t^T and π_t converges to 1 a.s. when k grows to infinity. Indeed, fast mean reversion and reduced long-run variance make Vasicek instantaneous rates behave as if they were constant and ρ_t^T collapses to π_t. To illustrate this point, in Figure 4 we plot the two prices for increasing values of k, while keeping $\theta = 0.0967$, $\sigma = 0.01$, $\xi = 0.2$ and $Y_0 = 0.02$ fixed.

5.1.4 Pricing kernel growth

We finally focus on the pricing kernel between s and t in case yields to maturity are affine. Then, the growth term of $M_{s,t}$ is an exponential function of Y_s, namely

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{A(s,T)\left(\frac{T-2s}{T-s}\right) - A(s,T-t) - \left\{B(s,T)\left(\frac{T-2s}{T-s}\right) - B(s,T-t)\right\}Y_s}.$$

When the terminal date T goes to infinity, this term converges a.s. to $e^{-r^\infty(t-s)}$, as expected. In Figure 5 we display this convergence under our Vasicek specifications by setting $s = 0$

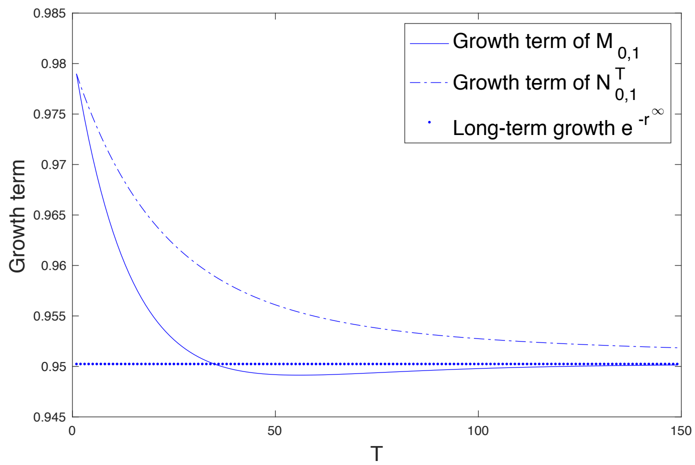


Figure 5: Term structure of growth terms of the pricing kernel $M_{0,1}$ and the rated-adjusted pricing kernel $N_{0,1}^T$ in Vasicek model. The solid line represents the growth term of $M_{0,1}$ for increasing horizons T, while the dashed line regards the growth term of $N_{0,1}^T$. The horizontal line is the long-term growth e^{-r^∞}.

and $t = 1$. In addition, we graphically compare the growth term of $M_{0,1}$ with the one of the rate-adjusted pricing kernel $N_{0,1}^T$. The two terms share the same long-run convergence.

5.2 Pricing with stocks and bonds

The situation is more involved when the market is generated by a set of risky assets with prices $X_t^1, \ldots X_t^N$, beyond pure discount bonds. Many sources of randomness are present, other than short-term rates. Although all the drift coefficients of no-arbitrage prices of such securities coincide with Y_t under Q, price dynamics become awkward when the forward measure is employed. The drifts of these prices under F^T are additionally affected by the correlation between the instantaneous rate and the idiosyncratic random component of the asset under consideration. Rabinovitch (1989) – that incorporates the previous Merton (1973) – discusses this point at length. On the contrary, the drift of the corresponding rate-adjusted prices is always equal to the yield r_s^T of zero-coupon bonds, for any traded security. To further elucidate the issue, we borrow the dynamics of rates and stock prices from Appendix B of Brigo and Mercurio (2006).

We assume that short-term rates move as in Vasicek model, with dynamics described by eq. (25) in the time interval $[s, T]$. Then, we consider a stock price X_t that follows a geometric Brownian motion with volatility $\eta > 0$, correlated with interest rates shocks. The instantaneous correlation parameter between the two underlying Wiener processes is ϕ. We can orthogonalize the two sources of randomness and consider, without loss of generality, the dynamics

$$\begin{cases} dX_t = X_t Y_t \; dt + \eta X_t \left(\phi dW_t^Q + \sqrt{1-\phi^2} dZ_t^Q \right) \\ dY_t = k\left(\theta - Y_t\right) dt + \sigma dW_t^Q, \end{cases}$$

where W_t^Q and Z_t^Q are independent Wiener processes. Under the forward measure, we get

$$\begin{cases} dX_t = X_t \left(Y_t - \frac{\phi\sigma\eta}{k} \left(1 - e^{-k(T-t)}\right) \right) dt + \eta X_t \left(\phi dW_t^{F^T} + \sqrt{1-\phi^2} dZ_t^{F^T} \right) \\ dY_t = \left(k\left(\theta - Y_t\right) - \frac{\sigma^2}{k} \left(1 - e^{-k(T-t)}\right) \right) dt + \sigma dW_t^{F^T}. \end{cases}$$

From the differential of X_t it is apparent that the correlation parameter ϕ impacts on the drift of X_t under the measure F^T.

In the following subsections we derive the Feynman-Kaç partial differential equations satisfied by no-arbitrage and rate-adjusted prices in this market and we determine the dynamics of the related pricing kernels.

5.2.1 Feynman-Kaç partial differential equations

Consider now a contingent claim h_T whose no-arbitrage price $\pi_t(h_T)$ is a regular function of t, X_t and Y_t, so that Itô's Lemma applies. We obtain the Feynman-Kaç partial differential equation for $\pi_t(h_T)$ by setting the drift of the discounted price $e^{-\int_s^t Y_\tau d\tau}\pi_t(h_T)$ equal to zero under Q. This process is, indeed, a Q-martingale. By setting $X_t = x$ and $Y_t = y$, we get

$$\frac{\partial \pi}{\partial t} + xy\frac{\partial \pi}{\partial x} + k\left(\theta - y\right)\frac{\partial \pi}{\partial y} + \frac{\eta^2 x^2}{2}\frac{\partial^2 \pi}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2 \pi}{\partial y^2} + \phi\sigma\eta x \frac{\partial^2 \pi}{\partial x \partial y} = y\pi \tag{26}$$

with terminal condition $\pi_T(h_T) = h_T$. A detailed derivation is in Appendix A. Note the role of the correlation parameter ϕ in weighing the cross derivative of π. In case $\phi = 0$ and interest rates are constant, the usual Black-Scholes partial differential equation arises.

We now develop a similar analysis for the rate-adjusted price of h_T. Since the forward price process $e^{r_s^T(T-t)}\rho_t^T(s, h_T)$ is an F^T-martingale, we set its drift equal to zero under F^T. Therefore, we obtain the Feynman-Kaç partial differential equation for ρ^T:

$$\begin{aligned} \frac{\partial \rho^T}{\partial t} &+ x\left(y - \frac{\phi\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right)\frac{\partial \rho^T}{\partial x} + \left(k\left(\theta - y\right) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right)\frac{\partial \rho^T}{\partial y} \\ &+ \frac{\eta^2 x^2}{2}\frac{\partial^2 \rho^T}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial^2 \rho^T}{\partial y^2} + \phi\sigma\eta x\frac{\partial^2 \rho^T}{\partial x \partial y} = r_s^T \rho^T \end{aligned} \tag{27}$$

with $\rho_T^T = h_T$. See the derivation in Appendix A.

Observe the different right-hand sides in eq. (26) and (27). They contain the instantaneous rate y for π and the yield to maturity r_s^T for ρ^T. The coefficients of *spatial* first-order derivatives are also different but the dissimilarity reduces when the speed k increases. In addition, the coefficients of $\partial\pi/\partial x$ and $\partial\rho^T/\partial x$ coincide when the correlation is absent.

Hence, the disparity between π and ρ^T can also be seized through the solution of different partial differential equations that share the same parabolic structure.

5.2.2 Pricing kernel dynamics

We now explicitly establish the evolution of $M_{s,t}$ and $N^T_{s,t}$ in our market. Beyond the money market account, the prices of the assets that generate the market satisfy

$$\begin{cases} dX_t = X_t Y_t \; dt + \eta X_t \left(\phi dW_t^Q + \sqrt{1-\phi^2} dZ_t^Q \right) \\ d\pi_t\left(1_T\right) = \pi_t\left(1_T\right) Y_t dt - \pi_t\left(1_T\right) B(t,T)\sigma dW_t^Q, \end{cases}$$

where the dynamics of $\pi_t(1_T)$ are derived as in Subsection 5.1.1. At the same time, under the physical measure,

$$\begin{cases} dX_t = X_t \mu_t^X dt + \eta X_t \left(\phi dW_t^P + \sqrt{1-\phi^2} dZ_t^P \right) \\ d\pi_t\left(1_T\right) = \pi_t\left(1_T\right) \mu_t^P dt - \pi_t\left(1_T\right) B(t,T)\sigma dW_t^P, \end{cases}$$

where μ_t^X and μ_t^P are adapted processes. They are related to the drifts under Q via the bivariate process of *market price of risk* $[\nu_t^W, \nu_t^Z]'$ such that

$$\begin{bmatrix} dW_t^Q \\ dZ_t^Q \end{bmatrix} = \begin{bmatrix} \nu_t^W \\ \nu_t^Z \end{bmatrix} dt + \begin{bmatrix} dW_t^P \\ dZ_t^P \end{bmatrix}.$$

By assuming that $\mu_t^P = (1 - \xi B(t,T)\sigma) Y_t$ for some $\xi > 0$, we obtain

$$\nu_t^W = \xi Y_t, \qquad \nu_t^Z = \frac{\mu_t^X - Y_t - \eta\phi\nu_t^W}{\eta\sqrt{1-\phi^2}},$$

where ν_t^W is in line with the usual approach to Vasicek short-term rates.

Now consider the pricing kernel $M_{s,t}$. Since the processes defined by $M_{s,t}H_t$, where H_t is each of X_t, $\pi_t(1_T)$ and B_t, are (conditional) P-martingales in $[s,T]$, their drifts are null and the dynamics of $M_{s,t}$ turns out to be

$$dM_{s,t} = -Y_t M_{s,t} dt - \nu_t^W M_{s,t} dW_t^P - \nu_t^Z M_{s,t} dZ_t^P.$$

The differential of the rate-adjusted pricing kernel $N^T_{s,t}$ can be inferred from the multiplicative relation $N^T_{s,t} = \pi_t(1_T) e^{r_s^T (T-t)} M_{s,t}$ by applying Itô's product rule. As a result, we have

$$dN^T_{s,t} = -r_s^T N^T_{s,t} dt - (\nu_t^W + B(t,T)\sigma) N^T_{s,t} dW_t^P - \nu_t^Z N^T_{s,t} dZ_t^P.$$

As expected, the dynamics of $N^T_{s,t}$ coincide with the ones of $M_{s,t}$ when interest rates are constant. Indeed, σ is null and the yield r^T_s coincides with the short-term rate. In general, the drift of N^T_s is driven by r^T_s in agreement with Theorem 9, while the one of $M_{s,t}$ exploits the stochastic rate Y_t.

6 Conclusions

This paper generalizes to stochastic-rate markets a key property of risk-neutral pricing. Indeed, if interest rates are constant (and deterministic) over time, the instantaneous rate is both the principal eigenvalue in the return-rate relation and the stochastic discount factor growth rate. If rates are stochastic, this feature is satisfied by rate-adjusted prices that, in fact, are indistinguishable from no-arbitrage prices when rates are constant. In particular, zero-coupon bond yields replace instantaneous rates and the forward measure is employed instead of the risk-neutral one. Importantly, bond yields are able to capture the growth rate of rate-adjusted pricing kernels. This rate coincides with the one of the effective pricing kernel when the horizon under consideration is infinite. Hence, the dual role of bond yields reconciles short-term and long-term properties of pricing and sheds light on the aggregation of increasing risk-exposures over time.

Introducing further specific dynamics of interest rates may constitute an interesting avenue for future research, with the purposes of quantifying numerically the difference between rate-adjusted and no-arbitrage prices. In a broader context, it could also be desirable to characterize the evolution of short rates through exogenous factors that determine the information structure.

From a theoretical perspective, a callenge is to study the implications of random eigenvalues in the Perron-Frobenius theory that underlies the pricing kernel decomposition. Indeed, random dominating eigenvalues may be an indicator of non-deterministic steady states for the dynamics of the economic variable under scrutiny.

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A Additional theoretical issues

Technical assumptions

In the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and is left-continuous at T. Specifically, we mean that $\mathbb{F}$ is complete and right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \in [0, T)$, and $\mathcal{F}_T = \mathcal{F}_{T-}$.

In the whole paper, we identify random variables that coincide almost surely and we identify stochastic processes up to modifications. Moreover, we consider processes $u : [0, T] \times \Omega \to \mathbb{R}$ that are adapted on the given filtered probability space. This requirement is equivalent to progressively measurability up to modifications, as proved, for instance, by Proposition 1.12 in Karatzas and Shreve (2012).

Forward measures

From the seminal work of Margrabe (1978), the use of different numéraires has become quite a common practice in asset pricing, especially in option valuation. Such changes of unit of measurements have proved to be convenient especially when dealing with stochastic interest rates, as Geman (1989) and Jamshidian (1989) discuss. A comprehensive treatment is given by Geman, El Karoui, and Rochet (1995). Among the others, Girotto and Ortu (1996) provide a characterization of numéraire portfolios. The no-arbitrage discussion in Delbaen and Schachermayer (1995) is also remarkable.

Regarding the T-forward measure, the Radon-Nikodym derivative of F^T with respect to the risk-neutral measure Q is

$$F_T^T = \frac{e^{-\int_0^T Y_\tau d\tau}}{\mathbb{E}\left[L_T e^{-\int_0^T Y_\tau d\tau}\right]} = e^{r_0^T T - \int_0^T Y_\tau d\tau},$$

$$F_t^T = \mathbb{E}_t\left[L_{t,T} F_T^T\right] = e^{r_0^T T - r_t^T (T-t) - \int_0^t Y_\tau d\tau}$$

and we set $F_{t,T}^T = F_T^T / F_t^T$. The Radon-Nikodym derivative of F^T with respect to P is, then, $G_T^T = F_T^T L_T$ and, from $F_t^T = \mathbb{E}_t[L_{t,T} F_T^T]$, we have

$$G_t^T = \mathbb{E}_t\left[G_T^T\right] = \mathbb{E}_t\left[L_T F_T^T\right] = L_t F_t^T.$$

As for t-bond yields, their relation with T-bond yields is expressed by the following compounding rule.

Lemma 10 *For any $s \leqslant t \leqslant T$, we have* $e^{r_s^T (T-s)} = e^{r_s^t (t-s)} \mathbb{E}_s^{F^T}\left[e^{r_t^T (T-t)}\right]$.

Proof of Lemma 10. Left to the reader. ■

This simple result additionally shows that $\mathbb{E}_s[M_{s,t}] = e^{-r_s^t (t-s)}$.

Weak time-derivative in $[s, T]$

Consider the conditional space $L^1_s(\mathcal{F}_T) = \{f \in L^0(\mathcal{F}_T) : \mathbb{E}_s[|f|] \in L^0(\mathcal{F}_s)\}$. Cerreia-Vioglio, Kupper, Maccheroni, Marinacci, and Vogelpoth (2016) show that $L^1_s(\mathcal{F}_T)$ is an L^0-module with the multiplicative decomposition $L^1_s(\mathcal{F}_T) = L^0(\mathcal{F}_s)L^1(\mathcal{F}_T)$. Clearly, $L^1_s(\mathcal{F}_T)$ contains all functions f in $L^1(\mathcal{F}_T)$: in this case $\mathbb{E}_s[|f|] \in L^1(\mathcal{F}_s)$. In general, however, the conditional expectation is defined for random variables that are merely in $L^0(\mathcal{F}_T)$. See, for instance, Chapter II, §7 of Shiryaev (1996).

In $L^1_s(\mathcal{F}_T)$ we use the L^0-valued metric $d(f, g) = \mathbb{E}_s[|f - g|]$. Accordingly, we say that a stochastic process $u : [s, T] \to L^1_s(\mathcal{F}_T)$ is L^1_s-continuous if and only if, for all $t \in [s, T]$, $\mathbb{E}_s[|u_\tau - u_t|] \to 0$ a.s. when $\tau \to t$. This property is weaker than standard L^1-continuity.

Now consider the L^0-module $\mathcal{U}_s$. As a consequence of Tonelli's theorem, all processes in $\mathcal{U}_s$ are such that $\int_s^T \mathbb{E}_s[|u_\tau|]d\tau$ belongs to $L^0(\mathcal{F}_s)$, where the integral is computed trajectory by trajectory. In addition, $\mathcal{U}_s$ includes all conditional (or generalized) martingales.

We now focus on weak time-differentiability in $[s, T]$. The space $C^1_c((t, T), L^0(\mathcal{F}_s))$ employed in Definition 1 consists of functions $\varphi_s : [t, T] \to L^0(\mathcal{F}_s)$ that have compact support in (t, T) and are continuously differentiable over time in the following sense: there exists a continuous function $\psi : [t, T] \to L^0(\mathcal{F}_s)$ with compact support in (t, T) such that the pathwise integral $\int_t^\tau \psi(z)dz$ equals $\varphi_s(\tau)$ for all $\tau \in [t, T]$. For simplicity, we denote ψ by φ'_s.

The next proposition shows that the weak time-derivative in $[s, T]$ is unique, up to modifications.

Proposition 11 *Let $u \in \mathcal{U}_s$ be weakly time-differentiable in $[s, T]$. Then, the weak time-derivative of u in $[s, T]$ is unique.*

Proof of Proposition 11. Follow the proof of Proposition 2.2 in Marinacci and Severino (2018) by replacing the unconditional expectation with the conditional expectation with respect to $\mathcal{F}_s$, and the convergence in L^1 with that in L^1_s. ■

We finally prove that a weakly time-differentiable process has null weak time-derivative in $[s, T]$ if and only if it is a conditional martingale.

Proposition 12 *A process u belongs to $\mathcal{U}^1_s$ and has $\mathcal{D}u = 0$ if and only if it is a conditional martingale.*

Proof of Proposition 12. Suppose that u is a conditional martingale. Then, u belongs to $\mathcal{U}_s$. Moreover, fixed $t \in [s, T]$, for any $\varphi_s \in C^1_c((t, T), L^0(\mathcal{F}_s))$ and $A_t \in \mathcal{F}_t$,

$$\int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] \varphi'_s(\tau)d\tau = \int_t^T \mathbb{E}_s\left[u_t \mathbf{1}_{A_t}\right] \varphi'_s(\tau)d\tau = \mathbb{E}_s\left[u_t \mathbf{1}_{A_t}\right] \int_t^T \varphi'_s(\tau)d\tau = 0$$

because φ_s is in $C^1_c((t, T), L^0(\mathcal{F}_s))$. Hence, $w(t) = 0$ for any $t \in [s, T]$ satisfies the definition of weak time-derivative of u in $[s, T]$ and so $\mathcal{D}u = 0$.

Conversely, assume that $u \in \mathcal{U}^1_s$ has $\mathcal{D}u = 0$. First, u is adapted and any $u_\tau \in L^1_s(\mathcal{F}_\tau)$. Therefore, $\mathbb{E}_s[|u_\tau|] \in L^0(\mathcal{F}_s)$ for all $\tau \in [s, T]$. As a consequence, $\mathbb{E}_t[|u_\tau|] \in L^0(\mathcal{F}_t)$ for all $s \leqslant t \leqslant \tau \leqslant T$. Indeed, since $|u_\tau|$ is nonnegative, $\mathbb{E}_t[|u_\tau|]$ is always defined as $\mathcal{F}_t$-measurable extended random variable. However, if there existed a set $A_t \in \mathcal{F}_t$ such that $\mathbb{E}_t[|u_\tau|]\mathbf{1}_{A_t}$ equals infinity, then, taken any $B_s \in \mathcal{F}_s$ with non-empty $A_t \cap B_s$, $\mathbb{E}_s[|u_\tau|\mathbf{1}_{B_s}] \geqslant \mathbb{E}_s[\mathbb{E}_t[|u_\tau|]\mathbf{1}_{A_t\cap B_s}]$ which is infinite. This fact would contradict $u_\tau \in L^1_s(\mathcal{F}_\tau)$.

Hence, in order to prove that u is a conditional martingale, we are just left to show that u satisfies the martingale property. We begin with proving that, given $t \in [s,T]$, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take into consideration a continuous function $\eta : [t,T] \to \mathbb{R}$ with compact support in (t,T) such that $\int_t^T \eta(\tau)d\tau = 1$. Given a continuous function $\xi : [t,T] \to \mathbb{R}$ with compact support in (t,T), we define the function $k_\xi : [t,T] \to \mathbb{R}$ by $k_\xi(\sigma) = \xi(\sigma) - (\int_t^T \xi(\tau)d\tau)\eta(\sigma)$.

The function k_ξ is continuous with compact support in (t,T) and $\int_t^T k_\xi(\tau)d\tau = 0$. Thus, k_ξ has a primitive K_ξ that is continuous with compact support in (t,T). Since $K_\xi \in C^1_c((t,T),\mathbb{R})$, it is included in $C^1_c((t,T),L^0(\mathcal{F}_s))$ and so we use it as a test function in the definition of weak time-derivative of u in $[s,T]$. Since $\mathcal{D}u = 0$, for any $A_t \in \mathcal{F}_t$ the following holds:

$$
\begin{aligned}
0 &= \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\left(\xi(\sigma) - \left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma)\right)d\sigma \\
&= \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\xi(\sigma)d\sigma - \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\left(\int_t^T \xi(\tau)d\tau\right)\eta(\sigma)d\sigma \\
&= \int_t^T \mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right]\xi(\tau)d\tau - \int_t^T \left(\int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma\right)\xi(\tau)d\tau \\
&= \int_t^T \left(\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma\right)\xi(\tau)d\tau.
\end{aligned}
$$

By Lemma A.1 in the Appendix of Marinacci and Severino (2018), for a.e. $\tau \in [t,T]$

$$
\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\eta(\sigma)d\sigma.
$$

Then, since $\int_t^T \eta(\sigma)d\sigma = 1$, we get

$$
\int_t^T \left(\mathbb{E}_s\left[u_\tau \mathbf{1}_{A_t}\right] - \mathbb{E}_s\left[u_\sigma \mathbf{1}_{A_t}\right]\right)\eta(\sigma)d\sigma = 0.
$$

As the last equality is satisfied by any continuous function η with compact support in (t,T), it follows that, for a.e. $\sigma \in [t,T]$, $\mathbb{E}_s[u_\sigma \mathbf{1}_{A_t}] = \mathbb{E}_s[u_\tau \mathbf{1}_{A_t}]$ and so $\mathbb{E}_t[u_\sigma] = \mathbb{E}_t[u_\tau]$. Consequently, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$ and so $\mathbb{E}_t[u_\tau] = f_t$ for some $f_t \in L^1_s(\mathcal{F}_t)$.

u is L^1_s-right-continuous and so $\mathbb{E}_t[u_\tau]$ goes to u_t in L^1_s when $\tau \to t^+$. Since for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[u_\tau]$ coincides a.s. with f_t, which does not depend on τ, the uniqueness of the L^1_s-limit ensures that $f_t = u_t$. Therefore, for any $t \in [0,T]$ and for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[u_\tau] = u_t$.

The last property is actually satisfied by any $\tau \in [t,T]$. Indeed, fix any τ and consider a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,T]$ such that $\tau_i \to \tau^+$ and $\mathbb{E}_t[u_{\tau_i}] = u_t$. Since u is L^1_s-right-continuous, the L^1_s-limit of $\mathbb{E}_t[u_{\tau_i}]$ is $\mathbb{E}_t[u_\tau]$. Nevertheless, $\mathbb{E}_t[u_{\tau_i}] = u_t$ for all i and so, by uniqueness of the L^1_s-limit, $\mathbb{E}_t[u_\tau] = u_t$. ■

Derivation of Feynman-Kaç partial differential equations

We take into consideration a payoff h_T as in Subsection 5.2.1 and we apply Itô's product rule to the discounted no-arbitrage price $e^{-\int_s^t Y_\tau d\tau}\pi_t(h_T)$. By omitting negligible higher order terms, we get

$$d\left(e^{-\int_s^t Y_\tau d\tau}\pi_t\left(h_T\right)\right) = e^{-\int_s^t Y_\tau d\tau}\left(d\pi_t\left(h_T\right) - Y_t\pi_t\left(h_T\right)dt\right).$$

Moreover,

$$\begin{aligned} d\pi_t\left(h_T\right) = {} & \frac{\partial \pi}{\partial t}dt + \frac{\partial \pi}{\partial x}X_t\left(Y_t dt + \eta\phi dW_t^Q + \eta\sqrt{1-\phi^2}dZ_t^Q\right) \\ & + \frac{\partial \pi}{\partial y}\left(k\left(\theta - Y_t\right)dt + \sigma dW_t^Q\right) \\ & + \frac{\partial^2 \pi}{\partial x^2}\frac{\eta^2 X_t^2}{2}dt + \frac{\partial^2 \pi}{\partial y^2}\frac{\sigma^2}{2}dt + \frac{\partial^2 \pi}{\partial x \partial y}\phi\sigma\eta X_t dt. \end{aligned}$$

Since $e^{-\int_s^t Y_\tau d\tau}\pi_t(h_T)$ defines a Q-martingale, the drift of its differential is null. As a result, by setting $X_t = x$ and $Y_t = y$, we obtain the Feynman-Kaç partial differential equation for the no-arbitrage price, i.e. eq. (26).

Regarding the rate-adjusted price of h_T, we have

$$d\left(e^{r_s^T(T-t)}\rho_t^T\left(s, h_T\right)\right) = e^{r_s^T(T-t)}\left(d\rho_t^T\left(s, h_T\right) - r_s^T\rho_t^T dt\right)$$

and

$$\begin{aligned} d\rho_t^T\left(s, h_T\right) = {} & \frac{\partial \rho^T}{\partial t}dt + \frac{\partial \rho^T}{\partial x}X_t\Bigg(\left(Y_t - \frac{\phi\sigma\eta}{k}\left(1 - e^{-k(T-t)}\right)\right)dt \\ & + \eta\phi dW_t^{F^T} + \eta\sqrt{1-\phi^2}dZ_t^{F^T}\Bigg) \\ & + \frac{\partial \rho^T}{\partial y}\left(\left(k\left(\theta - Y_t\right) - \frac{\sigma^2}{k}\left(1 - e^{-k(T-t)}\right)\right)dt + \sigma dW_t^{F^T}\right) \\ & + \frac{\partial^2 \rho^T}{\partial x^2}\frac{\eta^2 X_t^2}{2}dt + \frac{\partial^2 \rho^T}{\partial y^2}\frac{\sigma^2}{2}dt + \frac{\partial^2 \rho^T}{\partial x \partial y}\phi\sigma\eta X_t dt. \end{aligned}$$

By setting to zero the drift of the differential of the forward price process $e^{r_s^T(T-t)}\rho_t^T(s, h_T)$, which is an F^T-martingale, we deduce the Feynman-Kaç partial differential equation for ρ^T, namely eq. (27).

B Proofs of Section 3

B.1 Proof of Theorem 2

(Existence) In order to show that $\rho^T \in \mathcal{U}_s^1$, we prove that $e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$ and is weakly time-differentiable in $[s, T]$.

First, for all $\tau \in [s,T]$, $e^{r_s^T T}\rho_\tau^T \in L_s^1(\mathcal{F}_\tau)$. Indeed, $|e^{r_s^T T}\rho_\tau^T|$ is nonnegative and so its conditional expectation at time s is an extended real random variable. However, $\mathbb{E}_s^{F^T}[|e^{r_s^T T}\rho_\tau^T|] \leqslant e^{r_s^T \tau}\mathbb{E}_s^{F^T}[|h_T|]$, which is in $L^0(\mathcal{F}_s)$. Thus, $\mathbb{E}_s^{F^T}[|e^{r_s^T T}\rho_\tau^T|]$ is in $L^0(\mathcal{F}_s)$.

Regarding L_s^1−continuity, we check that, for any $t \in [s,T)$, $\mathbb{E}_s^F[|e^{r_s^T T}\rho_\tau^T - e^{r_s^T T}\rho_t^T|]$ tends to zero when $\tau \to t^+$. We have

$$\begin{aligned}&\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T T}\rho_\tau^T - e^{r_s^T T}\rho_t^T\right|\right] = e^{r_s^T t}\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T(\tau-t)}\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\\ &\leqslant e^{r_s^T t}\left(\mathbb{E}_s^{F^T}\left[\left|e^{r_s^T(\tau-t)}\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_\tau^{F^T}[h_T]\right|\right] + \mathbb{E}_s^{F^T}\left[\left|\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\right)\\ &\leqslant e^{r_s^T t}\left(\left|e^{r_s^T(\tau-t)} - 1\right|\mathbb{E}_s^{F^T}[|h_T|] + \mathbb{E}_s^{F^T}\left[\left|\mathbb{E}_\tau^{F^T}[h_T] - \mathbb{E}_t^{F^T}[h_T]\right|\right]\right).\end{aligned}$$

In the last expression, both addends go to zero a.s. when τ approaches t^+. In particular, the convergence of the first one follows from the fact that almost every realization of r_s^T is a real number (fixed for the convergence). As to the second term, its convergence is ensured by Lévy's downward theorem (applied with $\mathcal{F}_s$ instead of the trivial sigma-algebra) that guarantees that $\mathbb{E}_\tau^{F^T}[h_T]$ goes in L_s^1 to $\mathbb{E}_{t^+}^{F^T}[h_T] = \mathbb{E}_t^{F^T}[h_T]$ when $\tau \to t^+$. Similarly, when $\tau \to T^-$, the convergence is due to Lévy's upward theorem. Therefore $e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$.

Now we look for the weak time-derivative in $[s,T]$ of $e^{r_s^T T}\rho^T$. We consider any $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$. Since indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$,

$$\begin{aligned}&-\int_t^T \mathbb{E}_s^{F^T}\left[e^{r_s^T T}\rho_\tau^T\mathbf{1}_{A_t}\right]\varphi_s'(\tau)d\tau = -\int_t^T e^{r_s^T\tau}\mathbb{E}_s^{F^T}[h_T\mathbf{1}_{A_t}]\varphi'(\tau)d\tau\\ &= -\mathbb{E}_s^{F^T}[h_T\mathbf{1}_{A_t}]\int_t^T e^{r_s^T\tau}\varphi_s'(\tau)d\tau = \mathbb{E}_s^{F^T}[h_T\mathbf{1}_{A_t}]\int_t^T r_s^T e^{r_s^T\tau}\varphi_s(\tau)d\tau\\ &= \int_t^T r_s^T\mathbb{E}_s^{F^T}\left[e^{r_s^T\tau}h_T\mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau = \int_t^T \mathbb{E}_s^{F^T}\left[r_s^T e^{r_s^T T}\rho_\tau^T\mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau.\end{aligned}$$

The integral of the function $\sigma \mapsto e^{r_s^T\sigma}\varphi_s'(\sigma)$ is computed pathwise in $L^0(\mathcal{F}_s)$, exploiting the compact support of φ_s.

Therefore, the candidate weak time-derivative in $[s,T]$ of $e^{r_s^T T}\rho^T$ is $r_s^T e^{r_s^T T}\rho^T$. Since $r_s^T e^{r_s^T T}\rho^T$ belongs to $\mathcal{U}_s$, we can claim that $\mathcal{D}\rho^T = r_s^T\rho^T$. Of course, $\rho_T^T = h_T$ and so $\rho^T \in \mathcal{U}_s^1$ solves problem (10).

(Uniqueness) Let $f^{(1)}, f^{(2)} \in \mathcal{U}_s^1$ be two solutions of problem (10) and define $z = f^{(1)} - f^{(2)} \in \mathcal{U}_s^1$. We have that $\mathcal{D}z = r_s^T z$ and $z_T = 0$. We now compute the weak time-derivative of $e^{-rt}z_t$ in $[s,T]$. Fix $t \in [s,T]$. For any $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$, consider the function

$$\theta \mapsto e^{-r_s^T\theta}r_s^T\varphi_s(\theta) - e^{-r_s^T\theta}\varphi_s'(\theta)$$

that takes values in $L^0(\mathcal{F}_s)$. By integrating pathwise, it follows that

$$\int_\tau^T\left(e^{-r_s^T\theta}r_s^T\varphi_s(\theta) - e^{-r_s^T\theta}\varphi_s'(\theta)\right)d\theta = e^{-r_s^T\tau}\varphi_s(\tau).$$

Hence, $e^{-r_s^T\tau}\varphi_s(\tau)$ belongs to $C_c^1((t,T),L^0(\mathcal{F}_s))$ and so we can use it as test function in the definition of weak time-derivative of z in $[s,T]$:

$$\int_t^T \mathbb{E}_s^{F^T}\left[\mathcal{D}z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau}\right]\varphi_s(\tau)d\tau = -\int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t}\right]\left(e^{-r_s^T\tau}\varphi_s'(\tau) - e^{-r_s^T\tau}r_s^T\varphi_s(\tau)\right)d\tau$$
$$= -\int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau}\right]\varphi_s'(\tau)d\tau + \int_t^T \mathbb{E}_s^{F^T}\left[z_\tau \mathbf{1}_{A_t} e^{-r_s^T\tau} r_s^T\right]\varphi_s(\tau)d\tau.$$

Consequently, the weak time-derivative of $e^{-r_s^T t}z_t$ in $[s,T]$ is $e^{-r_s^T t}\left(\mathcal{D}z_t - r_s^T z_t\right)$.

However this process is null. Therefore, $e^{-r_s^T t}z_t$ has null weak time-derivative in $[s,T]$. Hence, by Proposition 12, $e^{-r_s^T t}z_t$ is a conditional F^T-martingale and so, for any $t\in[s,T]$ and $\tau\in[t,T]$, we have $\mathbb{E}_t^{F^T}[z_\tau] = e^{r_s^T(\tau-t)}z_t$. When τ goes to T^-, we get that $\mathbb{E}_t^{F^T}[z_\tau]$ tends to $e^{r_s^T(T-t)}z_t$ pointwise.

In addition, z_τ converges to $z_T=0$ in L_s^1 as τ approaches T^- and so $\mathbb{E}_t^{F^T}[z_\tau]$ tends to zero in L_s^1. By uniqueness of the L_s^1-limit, $z_t=0$ for all $t\in[s,T]$. This proves uniqueness of the solution of problem (10).

B.2 Proof of Proposition 3

Fix any positive s and consider the limit in probability when T goes to infinity. The fact that $r_s^T \xrightarrow{P} r^\infty$ ensures that $e^{-r_s^T(T-t)} \xrightarrow{P} 0$ for all $s>0$ and $t>s$. Therefore, $e^{-r_s^T(T-t)} - e^{-r_t^T(T-t)} \xrightarrow{P} 0$ and so $(\rho_t^T(s,h_T)-\pi_t(h_T))/\mathbb{E}_t^{F^T}[h_T]$ tends to zero.

As for the second convergence, since $r_s^T \xrightarrow{P} r^\infty$ for all $s>0$ as T goes to infinity, the difference $r_s^T - r_t^T$ converges in probability to zero for all $s>0$ and $t>s$. In addition, for any positive ε,

$$P\left(\frac{\left|\log\rho_t^T(s,h_T)-\log\pi_t(h_T)\right|}{T-t} > \varepsilon\right) = P\left(\left|r_s^T - r_t^T\right| > \varepsilon\right)$$

and this quantity goes to zero because $r_s^T - r_t^T \xrightarrow{P} 0$ as T increases.

B.3 Proof of Proposition 4

By Lemma 10 in Appendix A, for any $t>s$

$$\mathbb{E}_s^{F^T}\left[\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)}\right] = e^{r_s^T(t-s)}\mathbb{E}_s^{F^T}\left[\frac{\pi_s(1_T)}{\pi_t(1_T)}\right] = e^{r_s^T(t-s)}\pi_s(1_t) \xrightarrow{P} e^{\left(r^\infty - r_s^t\right)(t-s)}$$

as T goes to infinity.

As to the second convergence, consider the expression

$$\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)} = e^{r_s^T(t-s)}\frac{\pi_s(1_T)}{\pi_t(1_T)} = e^{r_s^T(t-s)}\frac{\pi_0(1_{T-s})}{\pi_0(1_{T-t})}\frac{\pi_s(1_T)}{\pi_0(1_{T-s})}\frac{\pi_0(1_{T-t})}{\pi_t(1_T)}.$$

Qin and Linetsky (2017) ensure that, as T goes to infinity,

- $e^{r_s^T(t-s)}$ converges to $e^{r^\infty(t-s)}$ in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-s-(t-s)})$ converges to $e^{-r^\infty(t-s)}$ in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$ converges to b_s^∞ in the semimartingale topology of Émery (1979);
- $\pi_0(1_{T-s})/\pi_t(1_T)$ converges to $1/b_t^\infty$ in the semimartingale topology.

In the semimartingale topology the product of convergent processes converges to the product of the respective limit processes. Moreover, the convergence in the semimartingale topology necessarily entails the convergence in probability for any fixed t. Therefore, by Slutsky's theorem we can conclude that

$$\frac{\rho_t^T(s,h_T)}{\pi_t(h_T)} \xrightarrow{P} \frac{e^{r^\infty(t-s)}}{e^{r^\infty(t-s)}}\frac{b_s^\infty}{b_t^\infty} = \frac{b_s^\infty}{b_t^\infty}, \qquad T \to +\infty.$$

B.4 Proof of Theorem 5

Following the proof of Theorem 2, it is easy to establish that the unique solution in $\mathcal{U}_s^1$ is given by the process defined, at any instant t, by $e^{-r^\infty(T-t)}\mathbb{E}_t^{F^\infty}[h_T]$. The forward measure and the yield in $[s,T]$ need to be replaced by F^∞ and r^∞, respectively.

B.5 Proof of Proposition 6

Fix $t \in [s,\tau]$. As we will show in Proposition 7, when T goes to infinity, $G_{t,\tau}^T$ converges in probability to $G_{t,\tau}^\infty$. Therefore, $G_{t,\tau}^T h_\tau$ goes to $G_{t,\tau}^\infty h_\tau$ in probability. Since $G_{t,\tau}^T h_\tau$ is also convergent in L^1 and convergence in L^1 implies the one in probability, by uniqueness of the limit, $G_{t,\tau}^T h_\tau$ tends to $G_{t,\tau}^\infty h_\tau$ in L^1. Consequently,

$$\mathbb{E}_t^{F^T}[h_\tau] = \mathbb{E}_t\left[G_{t,\tau}^T h_\tau\right] \xrightarrow{L^1} \mathbb{E}_t\left[G_{t,\tau}^\infty h_\tau\right] = \mathbb{E}_t^{F^\infty}[h_\tau], \qquad T \to +\infty$$

and the convergence is also in probability. In addition, $e^{-r_s^T(\tau-t)} \xrightarrow{P} e^{-r^\infty(\tau-t)}$ and so, by the continuous mapping theorem, when T goes to infinity,

$$\rho_t^T(s,h_\tau) = e^{-r_s^T(\tau-t)}\mathbb{E}_t^{F^T}[h_\tau] \xrightarrow{P} e^{-r^\infty(\tau-t)}\mathbb{E}_t^{F^\infty}[h_\tau] = \rho_t^\infty(s,h_\tau).$$

C Proofs of Section 4

C.1 Proof of Proposition 7

We exploit the asymptotic results of Qin and Linetsky (2017). Since T goes to infinity, we assume that $T > t+s$ without loss of generality.

As for the first convergence, consider

$$\frac{e^{r_s^T s}\pi_s(1_T)}{\pi_s(1_{T-t})} = e^{r_s^T s}\frac{\pi_0(1_{T-s})}{\pi_0(1_{T-t-s})}\frac{\pi_s(1_T)}{\pi_0(1_{T-s})}\frac{\pi_0(1_{T-t-s})}{\pi_s(1_{T-t})}.$$

When T goes to infinity, we have

- $e^{r_s^T s}$ converges to $e^{r^\infty s}$ in probability;
- $\pi_0(1_{T-s})/\pi_0(1_{T-t-s})$ converges to $e^{-r^\infty t}$ in probability;
- $\pi_s(1_T)/\pi_0(1_{T-s})$ converges to b_s^∞ in the semimartingale topology;
- $\pi_0(1_{T-t-s})/\pi_s(1_{T-t})$ converges to $1/b_s^\infty$ in the semimartingale topology.

As a result, the first convergence of the statement obtains.

Similarly,

$$\frac{e^{-r_s^T s}\pi_s\left(1_{T-t}\right)}{\pi_t\left(1_T\right)} = e^{-r_s^T s}\frac{\pi_0\left(1_{T-t-s}\right)}{\pi_0\left(1_{T-t}\right)}\frac{\pi_s\left(1_{T-t}\right)}{\pi_0\left(1_{T-t-s}\right)}\frac{\pi_0\left(1_{T-t}\right)}{\pi_t\left(1_T\right)}$$

and, when T goes to infinity,

- $e^{-r_s^T s}$ converges to $e^{-r^\infty s}$ in probability;
- $\pi_0(1_{T-t-s})/\pi_0(1_{T-t})$ converges to $e^{r^\infty s}$ in probability;
- $\pi_s(1_{T-t})/\pi_0(1_{T-t-s})$ converges to b_s^∞ in the semimartingale topology;
- $\pi_0(1_{T-t})/\pi_t(1_T)$ converges to $1/b_t^\infty$ in the semimartingale topology.

Consequently, the second convergence of the proposition is established.

In addition, Qin and Linetsky (2017) prove that $M_{s,t} = e^{-r^\infty(t-s)}(b_s^\infty/b_t^\infty)G_{s,t}^\infty$. At finite horizons we have

$$G_{s,t}^T = \left(\frac{e^{r_s^T s}\pi_s\left(1_T\right)}{\pi_s\left(1_{T-t}\right)}\right)^{-1}\left(\frac{e^{-r_s^T s}\pi_s\left(1_{T-t}\right)}{\pi_t\left(1_T\right)}\right)^{-1} M_{s,t}.$$

Since the first factor converges in probability to $e^{r^\infty(t-s)}$ and the second one tends to b_t^∞/b_s^∞, we deduce that $G_{s,t}^T$ converges in probability to $G_{s,t}^\infty$ when T goes to infinity.

C.2 Proof of Proposition 8

From the decomposition of $M_{s,t}$, we have that

$$N_{s,t}^T = e^{\left(r_s^T - r_t^T\right)(T-t)}e^{-r^\infty(t-s)}\frac{b_s^\infty}{b_t^\infty}G_{s,t}^\infty.$$

The quantity $e^{(r_s^T - r_t^T)(T-t)}$ coincides with the ratio $\pi_t(h_T)/\rho_t^T(s, h_T)$ when an arbitrary payoff h_T is considered. Thus, by Proposition 4, $e^{(r_s^T - r_t^T)(T-t)}$ converges in probability to b_t^∞/b_s^∞ as T goes to infinity, ensuring the convergence of $N_{s,t}^T$.

C.3 Proof of Theorem 9

Problem (21). We study the properties of the process defined, at any time t, by $G_s^T e^{r_s^T s} N_{s,t}^T$ that coincides with $e^{-r_s^T t} G_t^T$. Its conditional expectation at time s under the measure P belongs to $L^0(\mathcal{F}_s)$. Moreover, L_s^1-right-continuity at t is due to the fact that

$$\mathbb{E}_s\left[\left|e^{-r_s^T\tau}G_\tau^T - e^{-r_s^T t}G_t^T\right|\right] \leqslant e^{-r_s^T t}\left(\left|e^{-r_s^T(\tau-t)}-1\right| G_s^T + \mathbb{E}_s\left[\left|\mathbb{E}_\tau\left[G_T^T\right]-\mathbb{E}_t\left[G_T^T\right]\right|\right]\right)$$

for all $\tau \geqslant t$. Similarly to the proof of Theorem 2, Lévy's downward theorem ensures the convergence to zero when $\tau \to t^+$. A parallel reasoning guarantees L_s^1-left-continuity at T and so $e^{-r_s^T t}G_t^T$ belongs to $\mathcal{U}_s$.

Next, we show that the weak time-derivative in $[s,T]$ of $e^{-r_s^T t}G_t^T$ is $-r_s^T e^{-r_s^T t}G_t^T$ under the physical measure. By considering any $A_t \in \mathcal{F}_t$ and $\varphi_s \in C_c^1((t,T), L^0(\mathcal{F}_s))$, we have

$$\begin{aligned}
&-\int_t^T \mathbb{E}_s\left[e^{-r_s^T\tau}G_\tau^T \mathbf{1}_{A_t}\right]\varphi_s'(\tau)d\tau = -\mathbb{E}_s\left[G_T^T\mathbf{1}_{A_t}\right]\int_t^T e^{-r_s^T\tau}\varphi_s'(\tau)d\tau \\
&= -\mathbb{E}_s\left[G_T^T\mathbf{1}_{A_t}\right]\int_t^T r_s^T e^{-r_s^T\tau}\varphi_s(\tau)d\tau = \int_t^T \mathbb{E}_s\left[-r_s^T e^{-r_s^T\tau}G_\tau^T\mathbf{1}_{A_t}\right]\varphi_s(\tau)d\tau.
\end{aligned}$$

Moving back to $N_{s,t}^T$, we established that $N_{s,t}^T$ belongs to $\mathcal{U}_s^1$ and $\mathcal{D}N_{s,t}^T = -r_s^T N_{s,t}^T$.

Problem (22). It is convenient to consider the process defined, at any t, by $e^{r^\infty s}G_s^\infty N_{s,t}^\infty$, that is $e^{-r^\infty t}G_t^\infty$. Since G_t^∞ is a martingale under P, $\mathbb{E}_s[e^{-r^\infty t}G_t^\infty] = e^{-r^\infty t}G_s^\infty$ belongs to $L^0(\mathcal{F}_s)$. In addition, $\int_s^{+\infty}\mathbb{E}_s[e^{-r^\infty\tau}G_\tau^\infty]d\tau = G_s^\infty e^{-r^\infty s}/r^\infty$ is in $L^0(\mathcal{F}_s)$, too.

L_s^1-right-continuity at any t can be shown as in the proof of problem (21) by observing that $G_\tau^\infty = \mathbb{E}_\tau[G_T^\infty]$ and $G_t^\infty = \mathbb{E}_t[G_T^\infty]$ for any T larger than τ. As a result, $e^{-r^\infty t}G_t^\infty$ is in $\mathcal{U}_s$ with $T = +\infty$.

Regarding weak time-differentiability in $[s,+\infty)$, we can follow again the proof of problem (21) by integrating on intervals $[t,+\infty)$. Indeed, it is enough to use the relation $G_\tau^\infty = \mathbb{E}_\tau[G_T^\infty]$, where T is a time index larger than any instant in the (bounded) supports of φ_s and φ_s'. Thus, the weak time-derivative in $[s,+\infty)$ of $e^{-r^\infty t}G_t^\infty$ is $-r^\infty e^{-r^\infty t}G_t^\infty$.

Consequently, $N_{s,t}^\infty$ turns out to be a process in $\mathcal{U}_s^1$ with $T = +\infty$ that satisfies $\mathcal{D}N_{s,t}^\infty = -r^\infty N_{s,t}^\infty$.

D Cox, Ingersoll and Ross (1985) dynamics

We enhance the discussion of Subsection 5.1.1 about affine interest rate models by illustrating our theory when short-term rates follow Cox, Ingersoll, and Ross (1985) dynamics. The coefficients of the instantaneous rate are

$$\mu(t,Y_t) = k\theta - (k-\sigma\xi)Y_t, \qquad \sigma(t,Y_t) = \sigma\sqrt{Y_t}$$

with $k,\theta,\sigma,\xi > 0$ and $\sigma^2 < 2k\theta$. Here, the market price of risk is supposed to be $\xi\sqrt{Y_t}$. Under the risk-neutral measure Q, we get the dynamics

$$dY_t = k\left[\theta - Y_t\right]dt + \sigma\sqrt{Y_t}dW_t^Q. \tag{28}$$

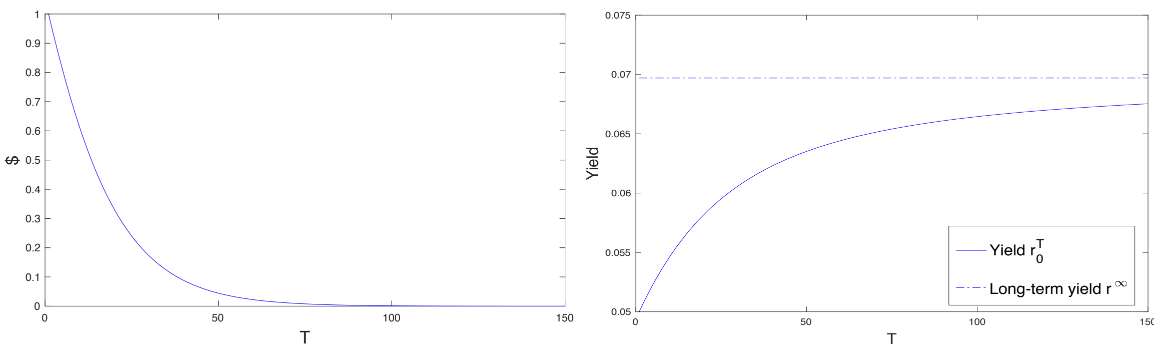
This specification ensures that interest rates are positive and their volatility raises when they increase. Moreover,

$$A(t,T) = \log\left(\left(\frac{2\sqrt{k^2+2\sigma^2}\exp\left(\left(k+\sqrt{k^2+2\sigma^2}\right)(T-t)/2\right)}{2\sqrt{k^2+2\sigma^2}+\left(k+\sqrt{k^2+2\sigma^2}\right)\left(\exp\left((T-t)\sqrt{k^2+2\sigma^2}\right)-1\right)}\right)^{2k\theta/\sigma^2}\right)$$

and

$$B(t,T) = \frac{2\left(\exp\left((T-t)\sqrt{k^2+2\sigma^2}\right)-1\right)}{2\sqrt{k^2+2\sigma^2}+\left(k+\sqrt{k^2+2\sigma^2}\right)\left(\exp\left((T-t)\sqrt{k^2+2\sigma^2}\right)-1\right)}$$

as mentioned in Section 3.2 of Brigo and Mercurio (2006).



(a) Zero-coupon bond prices term structure. (b) Yield-to-maturity term structure.

Figure 6: The left and right panel depict no-arbitrage pure discount bond prices (at time zero) and yields to maturity with respect to increasing horizons T under Cox, Ingersoll and Ross specifications. The dashed horizontal line represents the long-term yield r^∞.

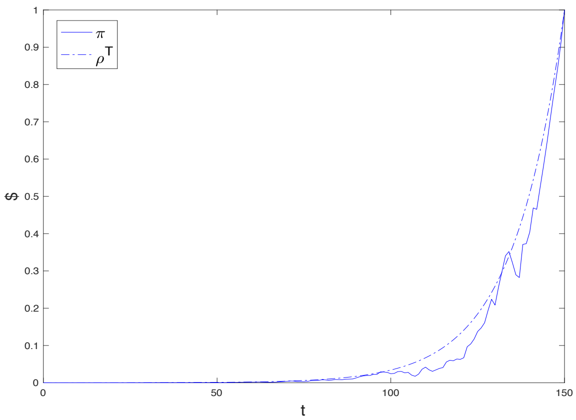
Under the forward measure the drift coefficient of the risk-neutral bond price turns out to be $(1+\sigma^2 B^2(t,T))Y_t$ and converges a.s. to

$$\left(1+\sigma^2\frac{4}{\left(k+\sqrt{k^2+2\sigma^2}\right)^2}\right)Y_t$$

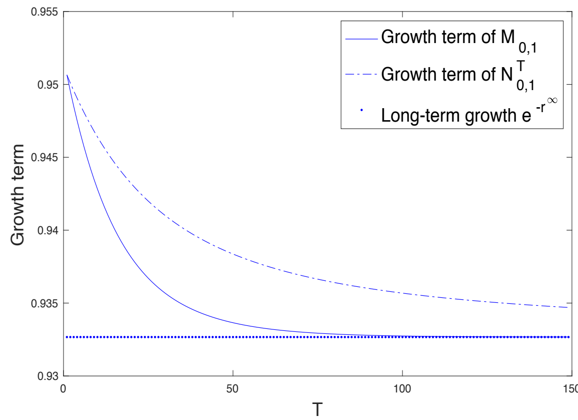
when the trading horizon T becomes infinitely large. Differently, the drift parameter r_0^T of the price $\rho_t^T(0,1_T)$ converges a.s. to the long-term yield $r^\infty = k\theta(\sqrt{k^2+2\sigma^2}-k)/\sigma^2$.

We plot in Figure 6 the term structures of no-arbitrage prices and yields of a zero-coupon bond in a simulated Cox, Ingersoll and Ross model, starting from time zero. We set the parameters $k = 0.0332$, $\theta = 0.0874$, $\sigma = 0.0265$ following Table III in Nowman (1997). We also fix $\xi = 0.2$ and $Y_0 = 0.05$. When maturity T becomes large, we notice that yields to maturity converge to the long-term yield r^∞, as expected. In the left panel of Figure 7, on the contrary, we fix T and we consider prices $\pi_t(1_T)$ and $\rho_t^T(0,1_T)$ of pure discount T-bonds.

As discussed in the body of the paper, $\pi_t(1_T)$ and $\rho_t^T(0, 1_T)$ are almost indistinguishable in the short-term as well as near maturity.



(a) Zero-coupon bond prices.



(b) Pricing kernel growth term structure.

Figure 7: On the left: values of $\rho_t^T(0, 1_T)$ and of a realization of $\pi_t(1_T)$ of a pure discount T-bond for any t in $[0, T]$. The two prices are represented by a dashed and a solid line, respectively. On the right: term structure of growth terms of the pricing kernel $M_{0,1}$ and the rate-adjusted pricing kernel $N_{0,1}^T$. The solid line represents the growth term of $M_{0,1}$ for increasing horizons T, while the dashed line regards the growth term of $N_{0,1}^T$. The horizontal line is the long-term growth e^{-r^∞}. All simulations assume Cox, Ingersoll and Ross dynamics for short rates.

Finally, we illustrate the growth term of the pricing kernel and of the rate-adjusted pricing kernel between time $s = 0$ and time $t = 1$. The right panel of Figure 7 depicts the convergence of both pricing kernel growth terms to $e^{-r^\infty(t-s)}$ when T goes to infinity.

E Application to optimal consumption

In the following we provide a comparison of the consequences of using rate-adjusted valuation for intertemporal consumption choices. We also highlight the role of the long-term yield in providing a threshold for subjective discounting that allows for well-defined consumption policies over infinite horizons.

We consider the complete market of Subsection 5.2. When T is finite, the market is generated by a risky asset, a pure discount T-bond and the money market account. When T is infinite, the zero-coupon bond is replaced by the long bond.

Based on this market, we consider two economies with an agent each. Every agent has subjective discount rate δ and time-separable quadratic expected utility on consumption. The first one, that we call *risk-neutral agent*, employs the pricing kernel $M_{s,t}$ for assessing the budget feasibility of consumption plans. Differently, the second agent, that we call *rate-adjusted*, uses $N_{s,t}^T$ for discounting future consumption.

We denote by w_s the $\mathcal{F}_s$-measurable initial endowment and by $c = \{c_t\}_{t\in[s,T]}$ any adapted consumption stream. The risk-neutral consumer maximizes her expected utility among budget-feasible consumption plans:

$$\max_{c} \quad \mathbb{E}_s\left[\int_s^T e^{-\delta(\tau-s)} u\left(c_\tau\right) d\tau\right] \tag{29}$$

$$\text{sub} \quad \mathbb{E}_s\left[\int_s^T M_{s,\tau} c_\tau d\tau\right] \leqslant w_s,$$

where $u(c_\tau) = -(c_\tau - b)^2/2$ and $b > 0$ is the bliss point. See, e.g. Cochrane (2014). We indicate by $c^* = \{c_t^*\}_{t\in[s,T]}$ the optimal policy of the risk-neutral agent.

On the contrary, the rate-adjusted consumer employs $N_{s,t}^T$ for discounting. By choosing among all feasible consumption plans, she solves the optimization problem

$$\max_{c} \quad \mathbb{E}_s\left[\int_s^T e^{-\delta(\tau-s)} u\left(c_\tau\right) d\tau\right] \tag{30}$$

$$\text{sub} \quad \mathbb{E}_s\left[\int_s^T N_{s,\tau}^T c_\tau d\tau\right] \leqslant w_s, \quad \mathbb{E}_s\left[\int_s^T M_{s,\tau} c_\tau d\tau\right] \leqslant w_s,$$

where the rate-adjusted valuation of future consumption streams adds to the budget feasibility constraint of problem (29). We denote the optimal consumption policy of the rate-adjusted agent by $\widehat{c}^*$.

c^* and $\widehat{c}^*$ may be different consumption plans, depending on the relevance of the rate-adjusted budget feasibility constraint in problem (30). In general, the rate-adjusted agent can never outperform the risk-neutral one because of the larger number of constraints to fulfill. However, there are intervals of the initial endowment in which the two consumers enjoy the same optimal policy. In Figure 8 we consider increasing values of the endowment. We show in simulation when the two optimal consumption streams are distinct and when they coincide. In the latter case, we also individuate the values of initial wealth that allow for optimal policies constantly equal to the bliss point. In all our simulations, we set $s = 0$ and $T = 75$, on monthly basis. For the Vasicek (1977) model we use the parameters $k = 0.0331$, $\theta = 0.0967$, $\sigma = 0.01$, $\xi = 0.2$ and $Y_0 = 0.07$. Moreover, we set $\phi = 0.5$, $\eta = 0.2$ and the drift coefficient of the risky asset $\mu_\tau^X = Y_\tau + 0.05$ for all $\tau \in [s,T]$. We fix the subjective discount rate $\delta = 0.001$, the bliss point $b = 3$ and we let the initial endowment vary from 1 to 50.

As a special case, we have that $c_\tau^* = \widehat{c}_\tau^* = b$ for all $\tau \in [s,T]$ if

$$w_s \geqslant b \max\left\{\mathbb{E}_s\left[\int_s^T M_{s,\tau} d\tau\right],\ \mathbb{E}_s\left[\int_s^T N_{s,\tau}^T d\tau\right]\right\}.$$

In this case, both consumers are very wealthy and they can afford a consumption plan constantly equal to the bliss point over the whole period. For these levels of initial endowment the rate-adjusted budget feasibility constraint turns out to be unnecessary. Moreover, in the previous threshold $\mathbb{E}_s[\int_s^T N_{s,\tau}^T d\tau] = (1 - e^{-r_s^T(T-s)})/r_s^T$ and this quantity converges to the inverse of r^∞ when T goes to infinity.

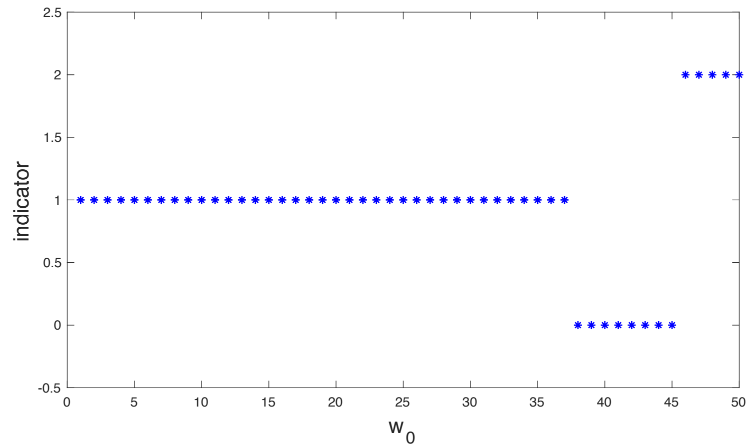
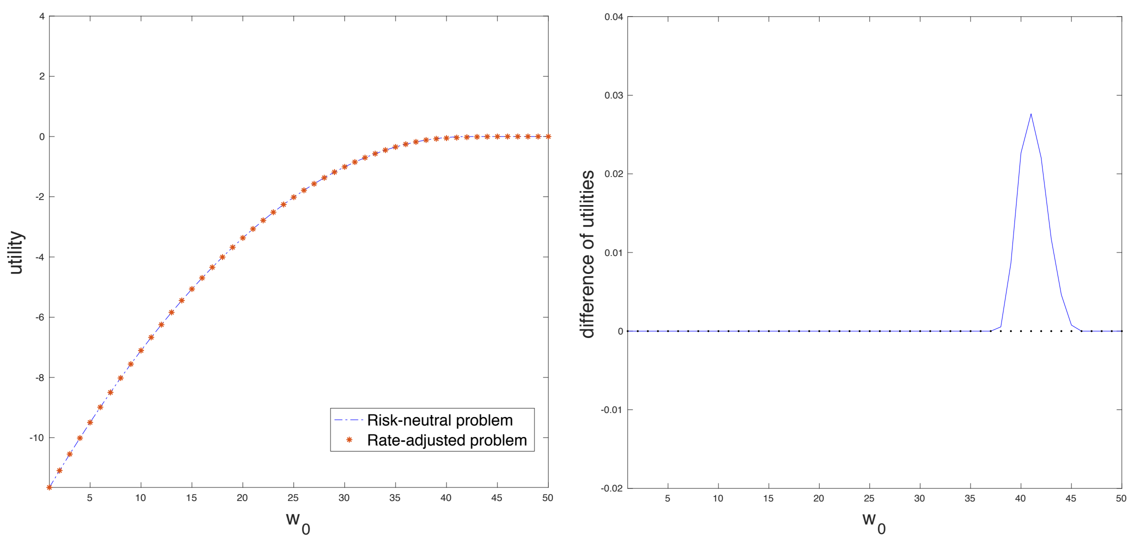


Figure 8: Across increasing initial endowments, the represented function equals 0 when optimal consumption policies are different in the two problems, 1 when optimal consumption policies coincide but do not reach the bliss point and 2 when they are both equal to the bliss point.



(a) Maximized expected utilities.

(b) Differences between utilities.

Figure 9: On the left, values of maximized expected utilities in problems (29) and (30) with respect to increasing initial endowments. On the right, difference of maximized utilities across increasing initial endowments.

In Figure 9 we plot the value of the maximized utilities and the difference between the indirect utilities in the two economies for increasing initial wealth. In our simulation, the difference between indirect utilities, when present, is tiny.

We provide the exact solutions of the risk-neutral and the rate-adjusted consumption problems in the next propositions.

Proposition 13 (Risk-neutral problem) *The solution of problem* (29) *is the following. For all $\tau \in [s,T]$,*

i) if $w_s \geqslant b\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$, then $c^*_\tau = b$;

ii) if $w_s < b\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$, then $c^*_\tau = b - e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$ with

$$\lambda_s = \frac{b\mathbb{E}_s\left[\int_s^T M_{s,\tau} d\tau\right] - w_s}{\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau\right]}.$$

Proof. The Lagrangian function associated with problem (29) is

$$\mathcal{L} = \mathbb{E}_s\left[\int_s^T e^{-\delta(\tau-s)} u\left(c_\tau\right) d\tau\right] - \lambda_s\left(\mathbb{E}_s\left[\int_s^T M_{s,\tau} c_\tau d\tau\right] - w_s\right),$$

where $\lambda_s \in L^0(\mathcal{F}_s)$ is nonnegative. By differentiating with respect to any realization of c_τ, we get the first order condition: $u'(c_\tau) = e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$. Since $(u')^{-1}(y) = b - y$, we obtain $c^*_\tau = b - e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$.

If $\lambda_s = 0$, then $c^*_\tau = b$ for all $\tau \in [s,T]$ and the constraint delivers the condition $w_s \geqslant b\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau]$. Hence, i) is proved.

If $\lambda_s > 0$, the constraint is binding and so λ_s takes the expression in ii). ■

Proposition 14 (Rate-adjusted problem) *The solution of problem* (30) *is the following. For all $\tau \in [s,T]$,*

i) if $w_s \geqslant b\max\{\mathbb{E}_s[\int_s^T M_{s,\tau} d\tau],\ \mathbb{E}_s[\int_s^T N^T_{s,\tau} d\tau]\}$, then $\widehat{c}^*_\tau = b$;

ii) if

$$\begin{aligned} &\left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N^T_{s,\tau}\right)^2 d\tau\right] - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau} N^T_{s,\tau} d\tau\right]\right) w_s \\ &< b\Bigg(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N^T_{s,\tau}\right)^2 d\tau\right]\mathbb{E}_s\left[\int_s^T M_{s,\tau}\right] \\ &\quad - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau} N^T_{s,\tau} d\tau\right]\mathbb{E}_s\left[\int_s^T N^T_{s,\tau} d\tau\right]\Bigg) \end{aligned}$$

and

$$\left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right] - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\right) w_s$$
$$< b\Bigg(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right]\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T\right]$$
$$- \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\mathbb{E}_s\left[\int_s^T M_{s,\tau}d\tau\right]\Bigg),$$

then $\widehat{c}_\tau^* = b - e^{\delta(\tau-s)}(\lambda_s M_{s,\tau} + \eta_s N_{s,\tau}^T)$ *with*

$$\lambda_s = \xi\Bigg\{b\Bigg(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right]\mathbb{E}_s\left[\int_s^T M_{s,\tau}d\tau\right]$$
$$- \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T d\tau\right]\Bigg)$$
$$- w_s\left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right] - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\right)\Bigg\},$$

$$\eta_s = \xi\Bigg\{b\Bigg(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right]\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T d\tau\right]$$
$$- \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\mathbb{E}_s\left[\int_s^T M_{s,\tau}d\tau\right]\Bigg)$$
$$- w_s\left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right] - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\right)\Bigg\},$$

$$\xi^{-1} = \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right]\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right] - \left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\right)^2.$$

iii) *if* $w_s < b\mathbb{E}_s[\int_s^T M_{s,\tau}d\tau]$ *and*

$$\left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right] - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\right) w_s$$
$$\geqslant b\Bigg(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right]\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T\right]$$
$$- \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}N_{s,\tau}^T d\tau\right]\mathbb{E}_s\left[\int_s^T M_{s,\tau}\right]\Bigg),$$

then $\widehat{c}_\tau^* = b - e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$ *with*

$$\lambda_s = \frac{b\mathbb{E}_s\left[\int_s^T M_{s,\tau}d\tau\right] - w_s}{\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}M_{s,\tau}^2 d\tau\right]};$$

iv) if $w_s < b\mathbb{E}_s[\int_s^T N_{s,\tau}^T d\tau]$ *and*

$$\left(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right] - \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau\right]\right) w_s$$
$$\geqslant b\Bigg(\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right]\mathbb{E}_s\left[\int_s^T M_{s,\tau}\right]$$
$$- \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau\right]\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T\right]\Bigg),$$

then $\widehat{c}_\tau^* = b - e^{\delta(\tau-s)}\eta_s N_{s,\tau}^T$ *with*

$$\eta_s = \frac{b\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T d\tau\right] - w_s}{\mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right]}.$$

Proof. The Lagrangian function related to problem (30) is

$$\mathcal{L} = \mathbb{E}_s\left[\int_s^T e^{-\delta(\tau-s)} u\left(c_\tau\right) d\tau\right] - \lambda_s\left(\mathbb{E}_s\left[\int_s^T M_{s,\tau} c_\tau d\tau\right] - w_s\right) - \eta_s\left(\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T c_\tau d\tau\right] - w_s\right)$$

with nonnegative $\lambda_s, \eta_s \in L^0(\mathcal{F}_s)$. The first order condition is $u'(c_\tau) = e^{\delta(\tau-s)}(\lambda_s M_{s,\tau} + \eta_s N_{s,\tau}^T)$ and so $c_\tau^* = b - e^{\delta(\tau-s)}(\lambda_s M_{s,\tau} + \eta_s N_{s,\tau}^T)$.

Case i) obtains when $\lambda_s = \eta_s = 0$. In this case, $c_\tau^* = b$ for all $\tau \in [s, T]$.

In ii) λ_s and η_s are positive and so their constraints are binding. Therefore, a linear system for λ_s and η_s arises:

$$\begin{cases} \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau}^2 d\tau\right]\lambda_s + \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau\right]\eta_s = b\mathbb{E}_s\left[\int_s^T M_{s,\tau} d\tau\right] - w_s \\ \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)} M_{s,\tau} N_{s,\tau}^T d\tau\right]\lambda_s + \mathbb{E}_s\left[\int_s^T e^{\delta(\tau-s)}\left(N_{s,\tau}^T\right)^2 d\tau\right]\eta_s = b\mathbb{E}_s\left[\int_s^T N_{s,\tau}^T d\tau\right] - w_s. \end{cases}$$

The unique solution is the one in the statement of ii). The conditions on the parameters in the claim ensure the positivity of λ_s and η_s.

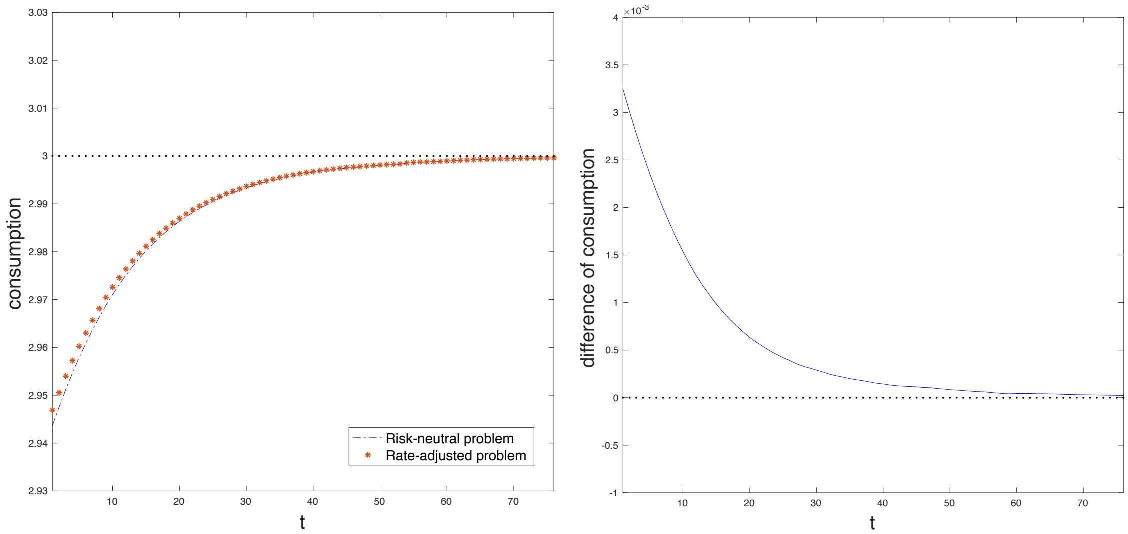
In case iii), λ_s is positive, while $\eta_s = 0$. Thus $c_\tau^* = b - e^{\delta(\tau-s)}\lambda_s M_{s,\tau}$. The feasibility constraint with $M_{s,\tau}$ is binding and so we can derive λ_s as in the statement. The constraint with $N_{s,\tau}^T$ delivers the inequality in the claim.

Case iv) is specular to iii). Just replace $M_{s,\tau}$ with $N_{s,\tau}^T$ and conversely. ■

In general, the ex-ante expected difference between the risk-neutral optimal policy and the rate-adjusted one is given by

$$\mathbb{E}_s\left[c_\tau^* - \widehat{c}_\tau^*\right] = \alpha_s e^{-(r_s^\tau - \delta)(\tau-s)} - \beta_s e^{-\left(r_s^T - \delta\right)(\tau-s)}$$

for some $\alpha_s, \beta_s \in L^0(\mathcal{F}_s)$. The difference depends on the relations between the yield r_s^T, the time-dependent yield r_s^τ and the subjective discount rate δ. In Figure 10 we consider a value of the endowment that delivers different optimal consumption streams. Then, we



(a) Average optimal consumption. (b) Difference between average optimal consumption.

Figure 10: On the left, average optimal consumption plans over time for a given initial endowment that delivers different optimal policies in problems (29) and (30). On the right, difference between the two average consumption policies.

depict the two expected optimal consumption plans and their difference, which turn out to be very small and vanishing over time in our simulations.

When the horizon is infinite, the rate-adjusted agent employs N^∞ as pricing kernel. Moreover, the relation between the long-term yield r^∞ and the subjective discount rate δ comes into play when we analyze the asymptotic behaviour of expected consumption over time.

On the one hand, if $\delta < r^\infty$ both $\mathbb{E}_s[c^*_\tau]$ and $\mathbb{E}_s[\widehat{c}^*_\tau]$ converge to the bliss point. The fact that the subjective discounting does not exceed the long-term yield constitutes a transversality condition that rephrases Corollary 1 in Ross (2015).

On the other hand, if $\delta > r^\infty$ both $\mathbb{E}_s[c^*_\tau]$ and $\mathbb{E}_s[\widehat{c}^*_\tau]$ fall. In this case of impatience, no finite optimal consumption plan can be found.

In general, the convergence rates of c^*_τ and $\widehat{c}^*_\tau$ satisfy the asymptotic relation

$$\frac{\log \mathbb{E}_s\left[b - c^*_\tau\right] - \log \mathbb{E}_s\left[b - \widehat{c}^*_\tau\right]}{\tau - s} \xrightarrow{P} 0, \qquad \tau \to +\infty,$$

that employs the distances of optimal policies from the bliss point.

Summing up, optimal consumption plans c^* and $\widehat{c}^*$ can be different but:

- when initial endowment is high enough, c^* and $\widehat{c}^*$ coincide;
- when the horizon is infinite and $\delta < r^\infty$, $\mathbb{E}_s[c^*_\tau]$ and $\mathbb{E}_s[\widehat{c}^*_\tau]$ coincide asymptotically for large τ.

Therefore, we can claim that expected consumption plans may differ just in the short term. Moreover, the long-term yield provides a threshold for the subjective discount rate in the long run.

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