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Weak time-derivatives and no arbitrage pricing

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Abstract We prove a risk-neutral pricing formula for a large class of semimartingale processes through a novel notion of weak time-differentiability that permits to differentiate adapted processes. In particular, the weak time-derivative isolates drifts of semimartingales and is null for martingales. Weak time-differentiability enables us to characterize no arbitrage prices as solutions of differential equations, where interest rates play a key role. Finally, we reformulate the eigenvalue problem of Hansen and Scheinkman [21] by employing weak time-derivatives.

Keywords no arbitrage pricing · weak time-derivative · martingale component · special semimartingales · stochastic interest rates

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1 Introduction

The no arbitrage pricing formula for the valuation of random payoffs is a milestone of asset pricing theory. For instance, it states that the price of a European option at time t is given by the conditional expectation of the discounted future payoff under a risk-neutral probability Q. If the option has payoff h_T at maturity T, a risk-neutral price π_t is

$$\pi_t = e^{-r(T-t)} \mathbb{E}_t^Q \left[h_T \right], \tag{1.1}$$

where $\mathbb{E}_t^Q$ denotes the conditional expectation under Q and r is a constant interest rate. The idea of pricing through risk-neutral expectations dates back to Bachelier [5] and Samuelson [33].

From eq. (1.1) it is clear that the instantaneous rate r plays a fundamental role in the option pricing. Indeed, price dynamics are determined by the same interest rate process, regardless of the option's terminal payoff. This intuition goes back to Cox and Ross [10], who derived the Black and Scholes [7] result by exploiting the accounting relations among bonds, stocks and options. This line of reasoning actually stems from the original approach of Modigliani and Miller [29].

In this paper we formalize the intuition that risk-neutral valuation is driven by the process of interest rates by generalizing to any risky asset the ordinary differential equation

$$\frac{dB}{dt} = rB$$

satisfied by the risk-free bond price B.

To obtain such a generalization, the issue of properly differentiating a random process arises. We address this problem by introducing the notion of *weak time-derivative* for adapted processes - denoted by $\mathcal{D}$ - and by showing the basic rules of its calculus. This differential notion allows us to prove that the no arbitrage pricing formula is the unique solution of the equation

$$\mathcal{D}\pi = r\pi \tag{1.2}$$

with the terminal condition $\pi_T = h_T$. We prove this result in Theorems 3.2 and 5.2, where we assume that interest rates are stochastic. This equation, which may be stated for a wide class of semimartingales (Proposition 3.1), formalizes the equality between the instantaneous return of a risky asset - namely $\mathcal{D}\pi/\pi$ - and the instantaneous interest rate r, which has to be valid in arbitrage-free markets.

The definition of weak time-derivative requires a suitable set of test functions (see Definitions 2.1 and 5.1) and involves the conditional expectation of the components of an adapted process. This notion provides a tractable characterization of martingales. Indeed, the weak time-derivative of a stochastic process is null if and only if the process is a martingale (Proposition 2.4). Hence, martingales play for weak time-derivatives the role that constant functions play in standard calculus.

The parallel between the calculus of weak time-derivatives - that we develop in Subsections 2.3 and 2.4 - and deterministic differential calculus is even deeper. For instance, the submartingale and supermartingale properties are related to monotonicity and can be established through the positive or negative sign of the weak time-derivative. In terms of interpretation, the weak time-derivative provides an indication of the upward or downward growth rate of the conditional expectation of the components of a random process.

Our approach permits to profit from analogies with standard differential analysis. In fact, the set of weakly time-differentiable processes coincides with a large class of special semimartingales and the weak time-derivative provides the derivative of the finite variation part (see Theorem 2.9 and Example 2.7). In other words, the weak time-derivative captures the drift of the semimartingale under consideration, a useful property when the canonical decomposition is unknown. In the financial application of Section 3 the identification of the drift and of the martingale component reminds of the logic behind Girsanov Theorem and is crucial for the analysis of eq. (1.2).

A nice feature of the weak time-derivative is that it applies to any adapted process, unlike the *infinitesimal generator* that requires the Feller property and the *extended infinitesimal generator* that requires a Markov process. Under these properties (and few regularity conditions) the weak time-derivative specializes to both these notions, as we illustrate in Corollaries 2.14 and 2.15. Indeed, the weak time-derivative allows us to deal with generalized formulations of problems that are usually formalized through generators.

As anticipated, the main results of the paper are summarized by Theorems 3.2 and 5.2, which show existence and uniqueness of the solution of eq. (1.2) under both deterministic and stochastic interest rates. In addition, Proposition 3.3 generalizes eq. (1.2) to cashflows.

When the risk-free rate is constant, by rewriting eq. (1.2) in operator form we obtain a reformulation of the eigenvalue-eigenvector problem analyzed by Hansen and Scheinkman [21] that employs the weak time-derivative in place of the extended infinitesimal generator. Following Hansen and Scheinkman, we obtain a decomposition of the stochastic discount factor into a martingale and a transient component in our more general setting.

Our paper combines different areas of mathematical analysis and stochastic calculus. The overall approach comes from variational calculus, it exploits the theory of Sobolev spaces and weak formulations of differential equations. See, for example, Lions [25], Adams and Fournier [1] and Brezis [8] for a comprehensive introduction to variational calculus, and Revuz and Yor [32] for stochastic calculus.

From a financial point of view, our work builds directly on the foundations of no arbitrage asset pricing theory illustrated, for instance, in Hansen and Richard [20], Delbaen and Schachermayer [13], Björk [6] and Föllmer and Schied [17]. In addition, our eigenvalue formulation refers to the long-term risk literature, in particular to Hansen and Scheinkman [21] and related works, like Alvarez and Jermann [3].

The paper is organized as follows. Section 2 develops the mathematical formalism of the weak time-derivative, while Sections 3 and 5 solve the no arbitrage pricing equation with deterministic and stochastic interest rates, respectively. A brief discussion of the special case of Black-Scholes model is presented in Subsection 3.3. Section 4 deals with the eigenvalue-eigenvector problem and the decomposition of the stochastic discount factor.

2 The weak time-derivative

After describing the setup, we define weak time-derivatives and illustrate their features. Then, we contrast weak time-derivatives with infinitesimal generators.

2.1 Setup

We fix a probability space $(\Omega, \mathcal{F}, P)$, a time interval $[0,T]$ and a dimension $N \in \mathbb{N}$. We consider a basic process $X = \{X_t\}_{t\in[0,T]} : [0,T] \times \Omega \to \mathbb{R}^N$ that generates a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$. In the financial application, $\mathbb{F}$ is the information structure available to agents. In particular, $X_t = [X_t^1, \dots, X_t^N]'$ for all $t \in [0,T]$ may be a bunch of primary asset prices at time t.

We assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfies the usual conditions, namely $\mathbb{F}$ is complete and right-continuous. Accordingly, we write $\mathcal{F}_t = \mathcal{F}_{t^+} = \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$. This property is satisfied, for instance, when X is a Lévy or a counting process or when X belongs to a large class of Markov processes (see Protter [31], Chapter I). In addition, we assume left-continuity at T, i.e. $\mathcal{F}_T = \mathcal{F}_{T^-}$.

Throughout the paper, random variables are identified almost surely. Also inequalities between random variables are meant almost surely. Moreover, we identify stochastic processes up to modifications.

On the filtered probability space, we consider processes $u : [0,T] \times \Omega \to \mathbb{R}$ that are adapted (and so are progressively measurable; see Proposition 1.12 in Karatzas and Shreve [23]).

As functions of time, our processes take values in $L^1(\mathcal{F}_T)$. With a small abuse of notation, we simply write $u : [0,T] \to L^1(\mathcal{F}_T)$. We will often require that u be L^1-continuous, meaning that, for every $t \in [0,T]$, $\mathbb{E}[|u_\tau - u_t|]$ tends to zero as τ goes to t.

We denote by $\mathcal{U}$ the space of adapted processes $u : [0,T] \to L^1(\mathcal{F}_T)$ that are L^1-right-continuous in $[0,T)$, L^1-left-continuous at T and have finite $\int_0^T \mathbb{E}[|u_\tau|]d\tau$.

As a result, any $u \in \mathcal{U}$ is Bochner integrable. Indeed, the finiteness of $\int_0^T \mathbb{E}[|u_\tau|]d\tau$ is necessary and sufficient for Bochner integrability of u. The Bochner integral of u is, then, an element of $L^1(\mathcal{F}_T)$ denoted by $\int_0^T u_\tau d\tau$. In addition, progressive measurability ensures the joint measurability of u on $[0,T] \times \Omega$. This property guarantees that the Bochner integral of u coincides almost surely with the pathwise Lebesgue integral (see Chapter II of Diestel and Uhl [14] and Section 11.8 of Aliprantis and Border [2]).

Observe that the space $\mathcal{U}$ contains all martingales defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Indeed, L^1-continuity is ensured by Lévy's Upward and Downward Theorems (see, for instance, Williams [35], Chapter 14).

Finally, $C([0,T])$ denotes the space of continuous functions $f:[0,T]\to\mathbb{R}$ and $C_c^n([0,T])$ the space of all n-times continuously differentiable functions $f:[0,T]\to\mathbb{R}$ with compact support.

2.2 Weak time-differentiability

We can now introduce the concept of *weak time-differentiability* for processes in the space $\mathcal{U}$.

Definition 2.1 We say that a process $u \in \mathcal{U}$ is *weakly time-differentiable* when there exists a process $v \in \mathcal{U}$ such that, for every $t \in [0,T]$,

$$\int_t^T \mathbb{E}\left[v_\tau \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau = -\int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau$$

for all $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^1([t,T])$. We call v a *weak time-derivative* of u.

Similarly, we say that a process $u \in \mathcal{U}$ is *twice weakly time-differentiable* when there exists a process $w \in \mathcal{U}$ such that, for every $t \in [0,T]$,

$$\int_t^T \mathbb{E}\left[w_\tau \mathbf{1}_{A_t}\right] \varphi(\tau) d\tau = \int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] \varphi''(\tau) d\tau$$

for all $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^2([t,T])$. We call w a *second-order weak time-derivative* of u.

This definition is well-posed because $u \in \mathcal{U}$ and so the integrals above are finite for any choice of $\varphi \in C_c^1([t,T])$ and of $A_t \in \mathcal{F}_t$.

The definition of weak time-derivative combines a variational approach, used for weak solutions of (deterministic) partial differential equations, with the information structure of the filtered probability space. Hence, the weak time-derivative builds a bridge between the *information-free* setting of calculus of variations and the adaptability concerns of the theory of stochastic processes. In fact, the presence of indicator functions reveals that weak time-differentiability actually involves the evolution of the conditional expectation of the components of a random process.

Definition 2.1 is stated in terms of the physical measure P. However, weak time-differentiability may be established also with respect to other probability measures. This is the case of asset pricing applications in which risk-neutral measures equivalent to P are employed for no arbitrage pricing, as illustrated in Section 3. Moreover, measure changes associated with the choice of a convenient numéraire are common practices in option pricing and interest rate theory. See, e.g., Margrabe [26] and Geman, El Karoui, and Rochet [19], among the others. The weak time-derivative may be a fruitful instrument also in these settings.

The weak time-derivative is unique, up to modifications.

Proposition 2.2 *A weakly time-differentiable $u \in \mathcal{U}$ has a unique weak time-derivative.*

Proof Let v and $\hat{v}$ be two weak time-derivatives of u. Then, for every $t \in [0,T]$

$$\int_t^T \mathbb{E}\left[(v_\tau - \hat{v}_\tau)\,\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau = 0$$

for any $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^1([t,T])$. By the lemma in Appendix, for a.e. $\tau \in [t,T]$, $\mathbb{E}[(v_\tau - \hat{v}_\tau)\mathbf{1}_{A_t}] = \mathbb{E}[0\mathbf{1}_{A_t}]$ for all $A_t \in \mathcal{F}_t$. Hence, the null process fits the definition of conditional expectation of $v_\tau - \hat{v}_\tau$ with respect to $\mathcal{F}_t$. Therefore, for a.e. $\tau \in [t,T]$, $\mathbb{E}_t[v_\tau] = \mathbb{E}_t[\hat{v}_\tau]$.

Now take into consideration a sequence $\{\tau_i\}_{i\in\mathbb{N}} \subset [t,\tau]$ such that $\tau_i \to t^+$ and $\mathbb{E}_t[v_{\tau_i}] = \mathbb{E}_t[\hat{v}_{\tau_i}]$ for all i. Since v is L^1-right-continuous, $\mathbb{E}_t[v_{\tau_i}]$ converges in L^1 to v_t as τ_i approaches t^+. Simultaneously, $\mathbb{E}_t[\hat{v}_{\tau_i}]$ converges in L^1 to $\hat{v}_t$. Since $\mathbb{E}_t[v_{\tau_i}]$ and $\mathbb{E}_t[\hat{v}_{\tau_i}]$ coincide a.s., by uniqueness of the L^1-limit, $v_t = \hat{v}_t$. □

We denote by $\mathcal{D}u$ the weak time-derivative of u. Moreover, we introduce the space

$$\mathcal{U}^1 = \{u \in \mathcal{U} :\ u \text{ is weakly time-differentiable}\},$$

endowed with the norm $\|u\|_{\mathcal{U}^1} = \int_0^T \mathbb{E}[|u_\tau|]d\tau + \int_0^T \mathbb{E}[|(\mathcal{D}u)_\tau|]d\tau$.

For deterministic processes, the weak time-derivative reduces to the classical derivative of calculus.

Proposition 2.3 *Let $g \in C([0,T])$ and $u_t = \int_0^t g(s)ds$ for all $t \in [0,T]$. Then, $\mathcal{D}u = g$.*

Proof As g is deterministic and continuous, g and u belong to $\mathcal{U}$. Taken any $A_t \in \mathcal{F}_t$ and any $\varphi \in C_c^1([t,T])$, for every $t \in [0,T]$ we have

$$\begin{aligned}\int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau &= \mathbb{E}\left[\mathbf{1}_{A_t}\right]\int_t^T\left(\int_0^\tau g(s)ds\right)\varphi'(\tau)d\tau \\ &= -\mathbb{E}\left[\mathbf{1}_{A_t}\right]\int_t^T g(\tau)\varphi(\tau)d\tau = -\int_t^T \mathbb{E}\left[g(\tau)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau\end{aligned}$$

because φ has compact support. Hence, g is the weak time-derivative of u. □

2.3 Calculus of the weak time-derivative

We start by establishing the equivalence between processes with null weak time-derivative and martingales.

Proposition 2.4 *A process u belongs to $\mathcal{U}^1$ and has $\mathcal{D}u = 0$ if and only if it is a martingale.*

Proof Assume that u is a martingale. As observed in Subsection 2.1, u belongs to $\mathcal{U}$. Fix $t \in [0,T]$. For all $\varphi \in C_c^1([t,T])$ and $A_t \in \mathcal{F}_t$,

$$\int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau = \int_t^T \mathbb{E}\left[u_t \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau = \mathbb{E}\left[u_t \mathbf{1}_{A_t}\right] \int_t^T \varphi'(\tau) d\tau = 0$$

because φ is a function in $C_c^1([t,T])$. As a result, $v_t = 0$ for all $t \in [0,T]$ satisfies the definition of weak time-derivative of u, i.e. $\mathcal{D}u = 0$.

Conversely, suppose that $u \in \mathcal{U}^1$ has $\mathcal{D}u = 0$. We first show that, given $t \in [0,T]$, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t,T]$.

Take a continuous function $\eta : [t,T] \to \mathbb{R}$ with compact support such that $\int_t^T \eta(\tau) d\tau = 1$. Given a continuous function $\xi : [t,T] \to \mathbb{R}$ with compact support, we define the function $k_\xi : [t,T] \to \mathbb{R}$ by

$$k_\xi(s) = \xi(s) - \left(\int_t^T \xi(\tau) d\tau\right) \eta(s). \tag{2.1}$$

The function k_ξ is continuous with compact support and $\int_t^T k_\xi(\tau) d\tau = 0$. Hence, k_ξ has a primitive K_ξ which is continuous with compact support. As $K_\xi \in C_c^1([t,T])$, we employ it as a test function in the definition of weak time-derivative of u. Since $\mathcal{D}u = 0$, for all $A_t \in \mathcal{F}_t$ we have

$$\begin{aligned}
0 &= \int_t^T \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right] \left(\xi(s) - \left(\int_t^T \xi(\tau) d\tau\right) \eta(s)\right) ds \\
&= \int_t^T \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right] \xi(s) ds - \int_t^T \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right] \left(\int_t^T \xi(\tau) d\tau\right) \eta(s) ds \\
&= \int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] \xi(\tau) d\tau - \int_t^T \left(\int_t^T \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right] \eta(s) ds\right) \xi(\tau) d\tau \\
&= \int_t^T \left(\mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] - \int_t^T \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right] \eta(s) ds\right) \xi(\tau) d\tau.
\end{aligned}$$

The lemma in Appendix ensures that, for a.e. $\tau \in [t,T]$,

$$\mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] = \int_t^T \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right] \eta(s) ds.$$

Since $\int_t^T \eta(s)ds = 1$, we can rewrite the last equation as

$$\int_t^T \left\{\mathbb{E}\left[u_\tau \mathbf{1}_{A_t}\right] - \mathbb{E}\left[u_s \mathbf{1}_{A_t}\right]\right\} \eta(s)ds = 0.$$

As the last equality holds for any continuous function η with compact support in $[t, T]$, we find that, for a.e. $s \in [t, T]$, $\mathbb{E}[u_s \mathbf{1}_{A_t}] = \mathbb{E}[u_\tau \mathbf{1}_{A_t}]$. Therefore, $\mathbb{E}_t[u_s] = \mathbb{E}_t[u_\tau]$. As a result, $\mathbb{E}_t[u_\tau]$ is not dependent on τ for a.e. $\tau \in [t, T]$ and we can say that $\mathbb{E}_t[u_\tau] = f_t$ for some $f_t \in L^1(\mathcal{F}_t)$.

Since u is L^1-right-continuous, when τ goes to t^+, $\mathbb{E}_t[u_\tau]$ converges to u_t in L^1. Moreover, since for a.e. $\tau \in [t, T]$, $\mathbb{E}_t[u_\tau]$ coincides a.s. with f_t, which is not dependent on τ, the uniqueness of the L^1-limit implies that $f_t = u_t$. Hence, for any $t \in [0, T]$, for a.e. $\tau \in [t, T]$, $\mathbb{E}_t[u_\tau] = u_t$.

This property is actually satisfied by all $\tau \in [t, T]$, as it can be easily shown by following a reasoning similar to that of Proposition 2.2. □

A simple corollary of Proposition 2.4 shows that, given a weak time-derivative v, all processes $u \in \mathcal{U}^1$ such that $\mathcal{D}u = v$ differ by a martingale. Note that process u can be regarded as the weak time-antiderivative of process v, which is thus unique up to a martingale.

Corollary 2.5 *Let $v \in \mathcal{U}$ be the weak time-derivative of $u \in \mathcal{U}^1$. Then, v is also the weak time-derivative of $\hat{u} \in \mathcal{U}^1$ if and only if $\hat{u} = u + m$, where m is a martingale*

Proof Assume that v is also the weak time-derivative of $\hat{u} \in \mathcal{U}^1$ and consider the process $m = \hat{u} - u \in \mathcal{U}^1$. The weak time-derivative of m is null and so, by Proposition 2.4, m is a martingale. The converse implication is immediate. □

Moreover, the following result holds.

Proposition 2.6 *Let u be defined, for all $t \in [0, T]$, by*

$$u_t = \int_0^t g_s ds + m_t,$$

where $g \in \mathcal{U}$ and m is a martingale. Then, u belongs to $\mathcal{U}^1$ and $\mathcal{D}u = g$.

Proof As $g \in \mathcal{U}$, it is Bochner integrable. Thus, for all $t \in [0, T]$, the process $G_t = \int_0^t g_s ds$ is well-defined and adapted. First, observe that

$$\int_0^T \mathbb{E}\left[|G_\tau|\right] d\tau \leqslant \int_0^T \mathbb{E}\left[\int_0^T |g_s|\, ds\right] d\tau = T \int_0^T \mathbb{E}\left[|g_s|\right] ds,$$

which is finite because $g \in \mathcal{U}$. The exchange between the order of expectation (which is a bounded operator) and Bochner integral is made possible by Lemma 11.45 in Aliprantis and Border [2].

Second, note that for any $t \in [0, T]$, $\mathbb{E}[|G_\tau - G_t|]$ tends to zero as τ approaches t, ensuring L^1-continuity. As a result, G belongs to $\mathcal{U}$.

We now show that $\mathcal{D}G = g$. Given $t \in [0,T]$, consider any $\varphi \in C_c^1([t,T])$ and $A_t \in \mathcal{F}_t$. We exchange the order of expectation and Bochner integral and, later, we apply Fubini's Theorem and exploit the compact support of φ:

$$\int_t^T \mathbb{E}\left[G_\tau \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau = \int_t^T \left(\int_0^\tau \mathbb{E}\left[g_s \mathbf{1}_{A_t} \varphi'(\tau)\right] ds \right) d\tau$$
$$= \int_0^t \left(\int_t^T \mathbb{E}\left[g_s \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau \right) ds + \int_t^T \left(\int_s^T \mathbb{E}\left[g_s \mathbf{1}_{A_t}\right] \varphi'(\tau) d\tau \right) ds$$
$$= \int_0^t \left(\mathbb{E}\left[g_s \mathbf{1}_{A_t}\right] \int_t^T \varphi'(\tau) d\tau \right) ds + \int_t^T \left(\mathbb{E}\left[g_s \mathbf{1}_{A_t}\right] \int_s^T \varphi'(\tau) d\tau \right) ds$$
$$= - \int_t^T \mathbb{E}\left[g_s \mathbf{1}_{A_t}\right] \varphi(s) ds.$$

So, g is the weak time-derivative of G.

As to m, this process belongs to $\mathcal{U}^1$ and $\mathcal{D}m = 0$ by Proposition 2.4. Therefore, by additivity, $u \in \mathcal{U}^1$ and $\mathcal{D}u = g$. □

Example 2.7 Assume that, for every $t \in [0,T]$,

$$u_t = \alpha t + m_t \tag{2.2}$$

with $\alpha \in \mathbb{R}$ and m a martingale. Then $\mathcal{D}u = \alpha$. In addition, by Corollary 2.5 all processes $u \in \mathcal{U}^1$ such that $\mathcal{D}u = \alpha$ may be written as in eq. (2.2). In other words, a process in $\mathcal{U}^1$ is the sum of a deterministic trend and a martingale if and only if it has constant weak time-derivative. In this case, the value of $\mathcal{D}u$ identifies the linear drift.

If $N = 1$ this feature may be retrieved, for instance, in the Black-Scholes model, in which the stock price satisfies $X_t = X_0 \exp((r - \sigma^2/2)t + \sigma W_t)$, where $r \in \mathbb{R}, \sigma > 0$ and W is a Wiener process under the risk-neutral measure Q. In fact, log prices are the sum of a deterministic drift and a martingale process, namely $\log(X_t) = \log(X_0) + (r - \sigma^2/2)t + \sigma W_t$. Here, the weak time-derivative of log prices captures the drift coefficient $r - \sigma^2/2$ under Q.

Example 2.8 When $N = 1$ another illustration of Proposition 2.6 comes from continuous Itô semimartingales (or generalized diffusions) like, for instance, the process X in $\mathcal{U}$ such that

$$X_t = X_0 + \int_0^t g_s ds + \int_0^t h_s dW_s,$$

where $g \in \mathcal{U}$, h is adapted and $\int_0^T \mathbb{E}[h_s^2] ds$ finite. The stochastic differential of the process is usually written as $dX_t = g_t dt + h_t dW_t$.

The Itô integral of h with respect to the Wiener process W is a martingale and it belongs to $\mathcal{U}^1$. As a result, the weak time-derivative of X is the drift g:

$$\mathcal{D}X = g.$$

Moreover, any process defined by $u_t = f(t, X_t)$ for all $t \in [0, T]$ is also a continuous Itô semimartingale if f is continuously differentiable with respect to t and twice continuously differentiable with respect to the second argument, say x. The stochastic differential of u can, then, be derived by Itô's formula and the weak time-derivative of u is identified as the drift in the stochastic differential of u:

$$\mathcal{D}u = g\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2}h^2\frac{\partial^2 f}{\partial x^2}.$$

Inspired by the last example, next we characterize weakly time-differentiable processes in terms of special semimartingales (see Chapter III, §7 in Protter [31]), a class of stochastic processes widely employed in dynamic asset pricing modelling.

Theorem 2.9 *A process $u \in \mathcal{U}$ is weakly time-differentiable if and only if it is a special martingale*

$$u = a + m,$$

with $a_t = \int_0^t g_s ds$, $a \in \mathcal{U}^1$, $g \in \mathcal{U}$ and m a martingale. In this case, $\mathcal{D}u = g$.

Proof Take $u \in \mathcal{U}^1$ and let $\mathcal{D}u$ denote its weak time-derivative. For every $t \in [0, T]$ define $a_t = \int_0^t (\mathcal{D}u)_s ds$. Hence, $a \in \mathcal{U}^1$ and it has the same weak time-derivative of u. Therefore, by Corollary 2.5, $m = u - a$ is a martingale. As a result, u decomposes into the sum $u = a + m$. Here, a has finite variation (because $\mathcal{D}u$ is integrable), it is càdlàg (because its paths are continuous) and adapted. Moreover, m is a local martingale (because it is a martingale). These features make u a semimartingale. In addition, a is also predictable (because it is left-continuous and adapted). As a result, u is a special semimartingale.

Conversely, let u be a special semimartingale as prescribed by the statement. By Proposition 2.6, u belongs to $\mathcal{U}^1$ and $\mathcal{D}u = g$. □

So, $\mathcal{U}^1$ is the space of special semimartingales that feature a (unique) absolutely continuous finite variation term and a (unique) local martingale term which is actually a martingale. The innovation of our approach relies on the fact that this class of processes is characterized via a differentiability condition that does not require the knowledge, ex-ante, of the canonical decomposition of the semimartingale into consideration. In other words, the weak time-derivative permits the identification of the drift term of semimartingales in $\mathcal{U}^1$ even when the canonical decomposition is not available.

In the next example we dig into the martingale term of Theorem 2.9 that, in general, combines a continuous martingale and a pure-jump martingale.

Example 2.10 Set $N = 1$ and consider on the time interval $[0, T]$ the process X driven by the dynamics

$$\frac{dX_t}{X_{t^-}} = \mu dt + \sigma dW_t + dH_t,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, W is a Wiener process and H is a compound Poisson process. In particular, $H_t = \sum_{k=1}^{N_t} z_k$, i.e. $dH_t = z_{N_{t^-}+1}\, dN_t$, where N is a

Poisson process with intensity λ and z_k are i.i.d. random variables that are independent of W and N, with $\mathbb{E}[z_k] = z$ and $z_k \geqslant -1$. This jump-diffusion process has been first used by Merton [27] for option pricing. A taxonomy of similar stochastic differential equations applied in asset valuation can be found, for instance, in Platen and Bruti-Liberati [30].

Although in general H is not a martingale, the *compensated* Poisson process $\hat{H}$ defined by $\hat{H}_t = Y_t - \lambda z t$ for all $t \in [0, T]$ is a martingale. Therefore, we can rewrite the dynamics of X as

$$\frac{dX_t}{X_{t^-}} = (\mu + \lambda z)dt + \sigma dW_t + d\hat{H}_t.$$

In the last formulation, the finite variation term - eventually captured by the weak time-derivative - is driven by the deterministic drift coefficient $\mu + \lambda z$. However, the martingale term is the sum of two process: a continuous martingale given by the Wiener process and a pure-jump martingale individuated by the compensated Poisson process.

As Example 2.10 suggests, the decomposition of Theorem 2.9 holds for a wide class of Lévy processes. Beyond Merton's model, such processes are extensively employed for modelling asset price dynamics. Relevant examples are provided by Carr, Geman, Madan, and Yor [9] and Kou [24].

2.4 Monotonicity and convexity

We now discuss the connection between monotonicity and the sign of the weak time-derivative. Interestingly, these features can be related to *submartingale* and *supermartingale* properties, respectively.

Proposition 2.11 *Let $u \in \mathcal{U}^1$. We have $\mathcal{D}u \geqslant 0$ if and only if u is a submartingale.*

Similarly, $\mathcal{D}u \geqslant 0$ if and only if u is a supermartingale.

Proof By Theorem 2.9, for every $t \in [0, T]$,

$$u_t = \int_0^t (\mathcal{D}u)_s \, ds + m_t,$$

where m is a martingale. Hence, for any $\tau \in [t, T]$,

$$\mathbb{E}_t [u_\tau] = u_t + \mathbb{E}_t \left[\int_t^\tau (\mathcal{D}u)_s \, ds \right].$$

If $\mathcal{D}u \geqslant 0$, then $\mathbb{E}_t [u_\tau] \geqslant u_t$ and u is a submartingale.

Conversely, if u is a submartingale, $\mathbb{E}_t \left[\int_t^\tau (\mathcal{D}u)_s \, ds \right] \geqslant 0$ for every $t \in [0, T]$ and $\tau \in [t, T]$. Therefore, $(\mathcal{D}u)_s \geqslant 0$ for a.e. $s \in [0, T]$. By L^1-continuity of u, we can conclude that $\mathcal{D}u \geqslant 0$. □

Next we focus on the increments of weak time-derivatives, i.e. we deal with convexity. The following result shows that a process $u \in \mathcal{U}^1$ satisfies a convexity property when $\mathcal{D}u$ is increasing, after taking the conditional expectation.

Proposition 2.12 *Let $u \in \mathcal{U}^1$. For all $t \in [0,T]$ and all τ_1, τ_2 such that $t \leqslant \tau_1 \leqslant \tau_2 \leqslant T$, we have*

$$\mathbb{E}_t\left[(\mathcal{D}u)_{\tau_1}\right] \leqslant \mathbb{E}_t\left[(\mathcal{D}u)_{\tau_2}\right]$$

if and only if, for all $t \in [0,T]$ and all τ_1, τ_2 with $t \leqslant \tau_1 \leqslant \tau_2 \leqslant T$, we have

$$\mathbb{E}_t\left[(\mathcal{D}u)_{\tau_1}\right] \leqslant \frac{\mathbb{E}_t\left[u_{\tau_2}\right] - \mathbb{E}_t\left[u_{\tau_1}\right]}{\tau_2 - \tau_1} \leqslant \mathbb{E}_t\left[(\mathcal{D}u)_{\tau_2}\right].$$

A dual result holds with $\geqslant$ instead of $\leqslant$.

Proof From the proof of Proposition 2.11, for every $t \in [0,T]$ and τ_1, τ_2 such that $t \leqslant \tau_1 \leqslant \tau_2 \leqslant T$, we have

$$\mathbb{E}_t\left[u_{\tau_2}\right] - \mathbb{E}_t\left[u_{\tau_1}\right] = \mathbb{E}_t\left[\int_{\tau_1}^{\tau_2} (\mathcal{D}u)_s \, ds\right] = \int_{\tau_1}^{\tau_2} \mathbb{E}_t\left[(\mathcal{D}u)_s\right] ds,$$

where the exchange between expectation and Bochner integral is made possible by Lemma 11.45 in Aliprantis and Border [2].

If $\mathbb{E}_t\left[(\mathcal{D}u)_{\tau_1}\right] \leqslant \mathbb{E}_t\left[(\mathcal{D}u)_{\tau_2}\right]$ for every t and $\tau_1 \leqslant \tau_2$, it follows that

$$\mathbb{E}_t\left[(\mathcal{D}u)_{\tau_1}\right](\tau_2 - \tau_1) \leqslant \mathbb{E}_t\left[u_{\tau_2}\right] - \mathbb{E}_t\left[u_{\tau_1}\right] \leqslant \mathbb{E}_t\left[(\mathcal{D}u)_{\tau_2}\right](\tau_2 - \tau_1),$$

as we wanted to prove.

Conversely, if the last inequality holds, $\mathbb{E}_t[(\mathcal{D}u)_{\tau_1}] \leqslant \mathbb{E}_t[(\mathcal{D}u)_{\tau_2}]$. □

Summing up, the calculus of weak time-derivatives generalizes in a natural way some key results of standard differential calculus.

2.5 Comparison with the infinitesimal generator

We now relate the notion of weak time-derivative with the one of *infinitesimal generator*, widely employed in stochastic calculus. A further comparison of the applications of both instruments in option pricing is in Section 4.

We begin by considering the difference quotients of conditional expectations. These quantities converge to the weak time-derivative.

Proposition 2.13 *Let $u \in \mathcal{U}^1$ and $t \in [0,T]$. If, for any $\tau \in [t,T]$, the quotient $\mathbb{E}_t[u_{\tau+h} - u_\tau]/h$ converges in L^1 when $h \to 0^+$, then*

$$\frac{\mathbb{E}_t\left[u_{t+h}\right] - u_t}{h} \xrightarrow{L^1} (\mathcal{D}u)_t \qquad h \to 0^+.$$

Proof We first show that, for a.e. $\tau \in [t,T]$,

$$\frac{\mathbb{E}_t\left[u_{\tau+h}-u_\tau\right]}{h} \xrightarrow{L^1} \mathbb{E}_t\left[(\mathcal{D}u)_\tau\right] \qquad h \to 0^+. \tag{2.3}$$

By following the same steps of the proof of Proposition 2.11 we find that, for a.e. $\tau, \hat{\tau} \in [t,T]$, for every $\mathcal{F}_t$-measurable set A_t, we have

$$\mathbb{E}\left[\mathbb{E}_t\left[u_{\hat{\tau}}-u_\tau\right]\mathbf{1}_{A_t}\right] = \int_\tau^{\hat{\tau}} \mathbb{E}\left[(\mathcal{D}u)_s\,\mathbf{1}_{A_t}\right] ds.$$

By setting $\hat{\tau} = \tau + h$ for some $h > 0$, we have

$$\mathbb{E}\left[\frac{\mathbb{E}_t\left[u_{\tau+h}-u_\tau\right]}{h}\mathbf{1}_{A_t}\right] = \frac{1}{h}\int_\tau^{\tau+h} \mathbb{E}\left[(\mathcal{D}u)_s\,\mathbf{1}_{A_t}\right] ds.$$

Now we take the limit as $h \to 0^+$. By the Lebesgue Differentiation Theorem, the right-hand side converges to $\mathbb{E}[(\mathcal{D}u)_\tau \mathbf{1}_{A_t}]$. Moreover, if w_τ denotes the $\mathcal{F}_t$-measurable L^1-limit of $\mathbb{E}_t[u_{\tau+h}-u_\tau]/h$, the left-hand side converges to $\mathbb{E}[w_\tau \mathbf{1}_{A_t}]$. Consequently, $\mathbb{E}[w_\tau \mathbf{1}_{A_t}] = \mathbb{E}[(\mathcal{D}u)_\tau \mathbf{1}_{A_t}]$ for every $\mathcal{F}_t$-measurable set A_t. Hence, by definition of conditional expectation, $w_\tau = \mathbb{E}_t[(\mathcal{D}u)_\tau]$. As a result, the convergence in eq. (2.3) is proved.

Now recall that, since u and $\mathcal{D}u$ are L^1-right-continuous, as $\tau \to t^+$, $\mathbb{E}_t[u_\tau]$ and $\mathbb{E}_t[(\mathcal{D}u)_\tau]$ converge in L^1 to u_t and $(\mathcal{D}u)_t$, respectively. Also, the convergence of $\mathbb{E}_t[u_{\tau+h}-u_\tau]/h$ ensures that $\mathbb{E}_t[u_{\tau+h}]$ tends to $\mathbb{E}_t[u_\tau]$ in L^1 when $h \to 0^+$ for any $\tau \in [t,T]$. In particular, for any fixed $h > 0$, we have that $\mathbb{E}_t[u_{\tau+h}]$ tends to $\mathbb{E}_t[u_{t+h}]$ in L^1 as $\tau \to t^+$ and

$$\frac{\mathbb{E}_t\left[u_{\tau+h}-u_\tau\right]}{h} \xrightarrow{L^1} \frac{\mathbb{E}_t\left[u_{t+h}\right]-u_t}{h} \qquad \tau \to t^+.$$

Putting things together, for any $\tau \in [t,T], h > 0$ we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u_{t+h}\right]-u_t}{h} - (\mathcal{D}u)_t\right|\right] \\
&\quad\leqslant \mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u_{t+h}\right]-u_t}{h} - \frac{\mathbb{E}_t\left[u_{\tau+h}-u_\tau\right]}{h}\right|\right] \\
&\qquad + \mathbb{E}\left[\left|-(\mathcal{D}u)_t + \mathbb{E}_t\left[(\mathcal{D}u)_\tau\right]\right|\right] + \mathbb{E}\left[\left|\frac{\mathbb{E}_t\left[u_{\tau+h}-u_\tau\right]}{h} - \mathbb{E}_t\left[(\mathcal{D}u)_\tau\right]\right|\right].
\end{aligned}$$

The previous convergences allow us to choose $\tau \in [t,T]$ so that the first two terms in the right-hand side are arbitrarily small and the convergence in eq. (2.3) allows us to choose h so that the last term is arbitrarily little. Hence, when $h \to 0^+$, $(\mathbb{E}_t[u_{t+h}]-u_t)/h$ converges to $(\mathcal{D}u)_t$ in L^1. □

Fixing $t \in [0,T]$, we can state Proposition 2.13 as

$$\mathbb{E}_t\left[u_\tau\right] - u_t - (\mathcal{D}u)_t\,(\tau - t) \xrightarrow{L^1} 0 \qquad \tau \to t^+,$$

which is a first-order expansion of $\mathbb{E}_t[u_\tau]$ in a right neighbourhood of t, with the limit taken in L^1.

As described in Chapter VII of Revuz and Yor [32], the infinitesimal generator of a Feller process X is the operator $\mathcal{A}$ that maps any continuous bounded function f belonging to the domain of $\mathcal{A}$ into the function $\mathcal{A}f$ such that

$$\mathcal{A}f\left(X_t\right)=\lim_{h\to 0^+}\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h}\qquad \forall t\in[0,T].$$

The limit, here, is in the uniform topology over all states $\omega\in\Omega$ and $\mathcal{A}f$ is continuous and bounded.

Proposition 2.13 shows that the weak time-derivative in $\mathcal{U}^1$ generally works as the infinitesimal generator with the limit $h\to 0^+$ taken in the L^1-norm, without requiring the Feller property of the underlying process. Next we show that the weak time-derivative and the infinitesimal generator coincide when they are both well-defined.

Corollary 2.14 *Given a Feller process X, let $u\in\mathcal{U}^1$ be such that, for every $t\in[0,T]$, $u_t=f(X_t)$ with f continuous and bounded in the domain of $\mathcal{A}$. Then,*

$$(\mathcal{D}u)_t=\mathcal{A}f\left(X_t\right)\qquad \forall t\in[0,T].$$

Proof The function f is continuous and bounded. Moreover, $\mathcal{A}f$ is continuous and bounded and $\mathcal{A}f(X_t)$ belongs to L^1. Since, for any $\tau\in[t,T]$, $(\mathbb{E}_\tau[f(X_{\tau+h})]-f(X_\tau))/h$ converges to $\mathcal{A}f(X_\tau)$ as $h\to 0^+$ in the uniform topology,

$$\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)-f\left(X_\tau\right)\right]}{h}\xrightarrow{L^1}\mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right]\qquad h\to 0^+. \tag{2.4}$$

Indeed, since f is in the domain of the infinitesimal generator $\mathcal{A}$, we can find an arbitrary small $\varepsilon>0$ such that

$$\begin{aligned}\left|\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right| &\leqslant \sup_{\omega\in\Omega}\left|\frac{\mathbb{E}_t\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}-\mathcal{A}f\left(X_\tau\right)\right|\\ &\quad+\left|\mathcal{A}f\left(X_\tau\right)\right|\\ &\leqslant \varepsilon+\left|\mathcal{A}f\left(X_\tau\right)\right|.\end{aligned}$$

By the Conditional Dominated Convergence Theorem,

$$\mathbb{E}_t\left[\frac{\mathbb{E}_\tau\left[f\left(X_{\tau+h}\right)\right]-f\left(X_\tau\right)}{h}\right]\xrightarrow{a.s.}\mathbb{E}_t\left[\mathcal{A}f\left(X_\tau\right)\right]$$

when $h\to 0^+$, that is $\mathbb{E}_t[f(X_{\tau+h})-f(X_\tau)]/h$ converges to $\mathbb{E}_t[\mathcal{A}f(X_\tau)]$ a.s. Moreover, $|\mathbb{E}_t[f(X_{\tau+h})-f(X_\tau)]/h|$ is bounded by $\varepsilon+\mathbb{E}_t[|\mathcal{A}f(X_\tau)|]$. Therefore, by the Dominated Convergence Theorem, for every $t\in[0,T]$ and $\tau\in[t,T]$, the convergence in (2.4) holds. In particular,

$$\frac{\mathbb{E}_t\left[f\left(X_{t+h}\right)\right]-f\left(X_t\right)}{h}\xrightarrow{L^1}\mathcal{A}f\left(X_t\right)\qquad h\to 0^+.$$

Since $(\mathbb{E}_t[f(X_{\tau+h})] - f(X_\tau))/h$ is convergent in L^1 when $h \to 0^+$ for every $t \in [0,T]$ and $\tau \in [t,T]$, Proposition 2.13 applies. In consequence, by uniqueness of the L^1-limit, $(\mathcal{D}u)_t = \mathcal{A}f(X_t)$. □

As we will see in Section 4, weak time-derivatives provide more general formulations of operator equations that are usually expressed through infinitesimal generators, such as the eigenvalue-eigenvector problem $\mathcal{A}f = rf$. This generalization is due to the fact that both instruments provide a similar characterization of martingales. Indeed, the process $\{f(X_t)\}_{t\in[0,T]}$ is a martingale when the infinitesimal generator of f is null, as ensured by Proposition 1.6 in Chapter VII of Revuz and Yor [32]. This result parallels Proposition 2.4 for weak time-derivatives.

2.6 Comparison with the extended infinitesimal generator

To illustrate the relation between the weak time-derivative and the extended infinitesimal generator, we begin with a corollary of Proposition 2.6.

Corollary 2.15 *Let $u \in \mathcal{U}^1$. Then, the process z defined, for all $t \in [0,T]$, by*

$$z_t = u_t - u_0 - \int_0^t (\mathcal{D}u)_\tau \, d\tau$$

is a martingale.

Proof z belongs to $\mathcal{U}$ as discussed in the proof of Proposition 2.6. Still by Proposition 2.6, the weak time-derivative of the process $U_t = \int_0^t (\mathcal{D}u)_s ds$ is $\mathcal{D}u$. Since the weak time-derivative of u_0 is null, by additivity we conclude that $\mathcal{D}z = \mathcal{D}u - 0 - \mathcal{D}u = 0$. Hence, by Proposition 2.4, z is a martingale. □

This result rephrases Dynkin's formula for Markov processes (see, e.g., Protter [31], Chapter II, §3). In particular, it ensures that, for all $t \in [0,T]$ and $\tau \in [t,T]$,

$$\mathbb{E}_t[u_\tau] = u_t + \mathbb{E}_t\left[\int_t^\tau (\mathcal{D}u)_s \, ds\right],$$

a fact that we already exploited in the proofs of Propositions 2.11 and 2.12.

In case X is a Markov process and $u_t = f(X_t)$ for all $t \in [0,T]$, the martingale property of z implied by Corollary 2.15 guarantees that the weak time-derivative of u coincides with the *extended infinitesimal generator* of f. Indeed, the extended infinitesimal generator of a measurable function f of X_t is a measurable function g such that $g(X_t)$ is integrable over time and the process z defined, for all $t \in [0,T]$, by $z_t = f(X_t) - f(X_0) - \int_0^t g(X_\tau)d\tau$ is a martingale (see Definition 1.8 in Chapter VII of Revuz and Yor [32]).

Corollary 2.16 *Let X_t be a Markov process and $u \in \mathcal{U}^1$ such that $u_t = f(X_t)$ for any t in $[0,T]$. Then, $\mathcal{D}u$ is the extended infinitesimal generator of f.*

Proof By Corollary 2.15, the process z defined by $z_t = u_t - u_0 - \int_0^t (\mathcal{D}u)_\tau d\tau$ is a martingale. Therefore, $\mathcal{D}u$ satisfies the definition of extended infinitesimal generator of f. □

Up to minor adjustments, the extended infinitesimal generator has been used by Hansen and Scheinkman [21] for the analysis of pricing operators. In fact, the extended infinitesimal generator keeps the most relevant features of the infinitesimal generator - in particular, the nullity for martingales - without requiring the Feller property of the underlying process.

In sum, the weak time-derivative is a general notion that permits the differentiation of adapted processes and that nicely compares with both the infinitesimal generator and the extended infinitesimal generator. Indeed, the infinitesimal generator applies to Feller processes and its extended version involves Markov processes, while the weak time-derivative is defined for the important class of special semimartingales. Moreover, the weak time-derivative mostly relies on measure theoretical assumptions, while the infinitesimal generator requires more restrictive topological conditions. Finally, it permits to rely more easily on *differential* intuitions.

3 No arbitrage pricing

3.1 Market setup

We consider a continuous-time market on $[0, T]$ with N risky assets, whose prices are collected in the vectorial process X, where $X_t = [X_t^1, \ldots, X_t^N]'$.

A risk-free security is also traded, with price B such that $B(t) = e^{rt}$ for all $t \in [0, T]$. Following for instance Björk [6], Chapter 10, for any $t \in [0, T]$ we define the vector of relative asset prices Z_t with $Z_t^i = X_t^i / B(t)$ for $i = 1, \ldots, N$. We then consider as environment the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is the filtration generated by Z and P is the physical probability, requiring that usual conditions hold.

We assume that our price system satisfies the no free lunch with vanishing risk (NFLVR) condition and that relative asset prices Z are semimartingales. The dynamics of these processes are, indeed, compatible with NFLVR restrictions, as explained by Delbaen and Schachermayer [12].

According to the First Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer [13], NFLVR ensures that there exists a, possibly not unique, probability measure Q equivalent to P such that Z is a *sigma-martingale*. In other words, Z is the martingale transform of some martingale via an integrable predictable process (see Émery [16]).

In the rest of the section we assume that at least one of the measures Q inferred by Delbaen and Schachermayer [13] is an *equivalent martingale measure*, i.e. it makes Z a martingale process. We consider a such measure Q for the valuation of securities in the market. In particular, we move to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. Integrability conditions, expectations and convergences are computed with respect to Q.

Since relative prices are martingales under Q, each Z^i belongs to $\mathcal{U}^1$. In what follows, we look for discounted price processes of attainable payoffs in this space of special semimartingales. Moreover, any process u belongs to $\mathcal{U}^1$ if and only if u/B belongs to $\mathcal{U}^1$, a fact that will be apparent from Proposition 3.1 and Theorem 3.2. Therefore, we concentrate directly on (non-discounted) payoff prices in $\mathcal{U}^1$. In general, special semimartingale prices are compatible with no free lunch conditions, as in Föllmer and Schweizer [18] and Ansel and Stricker [4], and they are included in the generic semimartingales required by Delbaen and Schachermayer [12] and Delbaen and Schachermayer [13].

3.2 Pricing

We consider a payoff h_T in $L^1(\mathcal{F}_T)$ associated, for instance, with a European option. We want to determine a price process π for h_T in $\mathcal{U}^1$ which is consistent with our arbitrage-free market, i.e. π/B is a martingale under Q.

Proposition 3.1 *Let Q be an equivalent martingale measure for Z and let $\pi \in \mathcal{U}^1$. The following conditions are equivalent:*

(i) *π is a no arbitrage price process;*
(ii) $\mathcal{D}(\pi/B) = 0$*;*
(iii) $\mathcal{D}\pi = r\pi$.

Proof We first prove $(i) \Leftrightarrow (ii)$ and then $(ii) \Leftrightarrow (iii)$.

$(i) \Leftrightarrow (ii)$ π is a no arbitrage price process if and only if π/B is a martingale under Q. By Proposition 2.4, this is equivalent to claim that the weak time-derivative of π/B is null.

$(ii) \Leftrightarrow (iii)$ For any $t \in [0,T]$, $\pi_t/B(t) = e^{-rt}\pi_t$. For any $A_t \in \mathcal{F}_t$ and $\varphi \in C_c^1([t,T])$,

$$\int_t^T \mathbb{E}^Q\left[(\mathcal{D}\pi)_\tau \mathbf{1}_{A_t}\right] e^{-r\tau}\varphi(\tau)d\tau = -\int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\right]\left(e^{-rt}\varphi(t)\right)'(\tau)d\tau$$
$$= -\int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\right] e^{-r\tau}\varphi'(\tau)d\tau + \int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\right] e^{-r\tau} r\varphi(\tau)d\tau.$$

As a result, the weak time-derivative of $e^{-rt}\pi_t$ is $e^{-rt}((\mathcal{D}\pi)_t - r\pi_t)$. Therefore, the weak time-derivative of π/B is null if and only if $\mathcal{D}\pi = r\pi$. □

This proposition may be viewed as a characterization of equivalent martingale measures. Indeed, Q is an equivalent martingale measure if and only if it is equivalent to P and, when used in the weak time-derivative, it ensures that $\mathcal{D}\pi = r\pi$ for all prices π in the market. This property is reminiscent of the outcome of Girsanov Theorem in Black-Scholes model (see Subsection 3.3), which makes the drift of the stock price proportional to r under Q.

When the price process is deterministic, point (iii) reduces to the usual differential equation solved by the price of a riskless bond. In particular, the bond price satisfies the boundary problem

$$\begin{cases} \frac{dB}{dt}(t) = rB(t) & t \in [0, T) \\ B(T) = e^{rT} \end{cases}$$

where the classical time-derivative is employed. In words, with continuous compounding, the rate of change of $B(t)$ is proportional to $B(t)$ and the coefficient of proportionality coincides with r.

Proposition 3.1 establishes a differential relation which is satisfied by the no arbitrage pricing function π of any payoff, risky or not. For instance, in the case of a European option we formulate the boundary problem

$$\begin{cases} (\mathcal{D}\pi)_t = r\pi_t & t \in [0, T) \\ \pi_T = h_T \end{cases} \tag{3.1}$$

where $h_T \in L^1(\mathcal{F}_T)$.

The financial interpretation of the problem is straightforward once we recall that $\mathcal{D}\pi$ is an adapted process and that the filtration $\mathbb{F}$ is right-continuous. Indeed, for any $t \in [0, T]$, the infinitesimal variation $(\mathcal{D}\pi)_t$ is known at time t and so the no arbitrage condition imposes that $\mathcal{D}\pi$ instantaneously behaves as the deterministic bond. The riskiness of π locally becomes immaterial. In other words, the rate of change of π_t must be proportional to π_t, as it is for the riskless asset price. Equivalently, instantaneous returns of h_T, i.e. $\mathcal{D}\pi/\pi$, coincide with the risk-free rate r when arbitrages are forbidden.

We now show that there exists a unique solution of Problem (3.1), given by the risk-neutral valuation formula of a payoff at T under the measure Q.

Theorem 3.2 *There exists a unique solution π of Problem* (3.1) *in $\mathcal{U}^1$, given by*

$$\pi_t = e^{-r(T-t)} \mathbb{E}_t^Q [h_T] \qquad \forall t \in [0, T]. \tag{3.2}$$

Proof Existence.

We prove that π belongs to $\mathcal{U}$ and it is weakly time-differentiable.

First, for all $\tau \in [0, T]$, $\pi_\tau \in L^1(\mathcal{F}_\tau)$ because $h_T \in L^1(\mathcal{F}_T)$. The integral of $\mathbb{E}^Q[|\pi_\tau|]$ over $[0, T]$ is finite, too.

As for the L^1-continuity, we check that, for any $t \in [0, T)$, $\mathbb{E}^Q[|\pi_\tau - \pi_t|]$ tends to zero as $\tau \to t^+$. By triangular inequality, $\mathbb{E}^Q[|\pi_\tau - \pi_t|]$ is smaller than

$$e^{-r(T-t)} \left\{ \left| e^{-r(t-\tau)} - 1 \right| \mathbb{E}^Q [|h_T|] + \mathbb{E}^Q \left[\left| \mathbb{E}_\tau^Q [h_T] - \mathbb{E}_t^Q [h_T] \right| \right] \right\}.$$

In the last expression, both addends go to zero as $\tau \to t^+$. In particular, the convergence of the second one is ensured by Lévy's Downward Theorem. A similar argument holds when $\tau \to T^-$. Therefore, π belongs to $\mathcal{U}$.

The fact that $r\pi$ satisfies the definition of weak time-derivative can be assessed by directly checking Definition 2.1. However, we follow an alternative path and define, for all $t \in [0,T]$, $\zeta_t = \mathbb{E}_t^Q[h_T]$ so that $\pi_t = e^{-r(T-t)}\zeta_t$. By Itô's differential rule, $d\pi_t = r\pi_t dt + e^{-r(T-t)}d\zeta_t$. By integrating in $[0,t]$ and employing integration by parts,

$$\pi_t - \pi_0 = \int_0^t r\pi_s ds + m_t, \qquad m_t = e^{-r(T-t)}\zeta_t - e^{-rT}\zeta_0 - r\int_0^t e^{-r(T-s)}\zeta_s ds,$$

where m_t defines a martingale. Since $r\pi$ belongs to $\mathcal{U}$, π is a special semi-martingale and, by Theorem 2.9, $\mathcal{D}\pi = r\pi$. Of course, $\pi_T = h_T$ and so $\pi \in \mathcal{U}^1$ solves Problem (3.1).

Uniqueness.
Let $\pi, \hat{\pi} \in \mathcal{U}^1$ be two solutions of Problem (3.1). By defining $z = \pi - \hat{\pi} \in \mathcal{U}^1$, we have that $\mathcal{D}z = rz$ and $z_T = 0$. As in the proof of Proposition 3.1, the weak time-derivative of $e^{-rt}z_t$ is $e^{-rt}((\mathcal{D}z)_t - rz_t)$. However this process is null. Therefore, by Proposition 2.4, $e^{-rt}z_t$ defines a martingale and so, for any $t \in [0,T]$ and $\tau \in [t,T]$

$$\mathbb{E}_t^Q[z_\tau] = e^{r(\tau-t)}z_t.$$

Letting τ go to T^-, $\mathbb{E}_t^Q[z_\tau]$ converges to $e^{r(T-t)}z_t$ pointwise. Simultaneously, z_τ converges to $z_T = 0$ in L^1 and so $\mathbb{E}_t^Q[z_\tau]$ tends to zero in L^1. By uniqueness of the L^1-limit, $z_t = 0$ for all $t \in [0,T]$. This proves uniqueness of the solution of Problem (3.1). □

To be consistent with the no arbitrage setting, Q must be an equivalent martingale measure for the extended market formed by the securities with discounted prices Z and π/B. Therefore, the only possible no arbitrage price process is given by eq. (3.2) because it satisfies

$$\frac{\pi_t}{B(t)} = \mathbb{E}_t^Q\left[\frac{h_T}{B(T)}\right]$$

for all $t \in [0,T]$. We refer to π as the *no arbitrage pricing function* (or *risk-neutral pricing function*) of any payoff h_T under Q.

The novelty of Theorem 3.2 relative to the literature consists in the fact that π belongs to the space $\mathcal{U}^1$ and is characterized by its dynamics $\mathcal{D}\pi = r\pi$.

3.3 Example: Black-Scholes model

The Black and Scholes [7] model involves a continuous-time financial market with a riskless bond with price B and a risky asset with price X. The bond and stock prices follow the dynamics

$$dB_t = rB_t dt, \qquad dX_t = \mu X_t dt + \sigma X_t d\overline{W}_t,$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility, $r \in \mathbb{R}$ is the risk-free rate and $\overline{W}$ is a P-Wiener process. The Girsanov Theorem ensures that there exists a probability measure Q equivalent to P under which the discounted stock price process is a martingale. According to the First Fundamental Theorem of Asset Pricing the market is arbitrage-free. In particular, the dynamics of the stock price under Q are

$$dX_t = rX_t dt + \sigma X_t dW_t,$$

where W is a Q-Wiener process. Hence, in this geometric Brownian motion setting, the risky security and the bond must share the same drift coefficient given by the interest rate r in order to exclude any arbitrage possibility. Specifically, the stock price $X_t = X_0 \exp((r - \sigma^2/2)t + \sigma W_t)$ is included in $\mathcal{U}$. Its integral representation is

$$X_t = X_0 + \int_0^t rX_s ds + \int_0^t \sigma X_s dW_s,$$

which individuates a continuous Itô semimartingale. Then, as in Example 2.8, the weak time-derivative captures the drift, that is $\mathcal{D}X = rX$. This relation turns out to be a restatement of the no arbitrage pricing equation of Problem (3.1) for the stock price under Q.

As described in the initial part of this section, the no arbitrage price process of a European option with payoff h_T is π given by eq. (3.2). In addition, in the Black-Scholes model π_t is a deterministic function of t and X_t. Since the discounted price process $\{e^{-rt}\pi_t\}_{t\in[0,T]}$ is also a Q-martingale, the drift of π is equal to r, too. This is the crucial outcome of no arbitrage, which is captured, more in general, by Problem (3.1), where no specific price dynamics are assumed. This is also the intuition that drives the Cox and Ross [10] derivation of the Black-Scholes equation, based on a hedging strategy that exploits a locally riskless portfolio.

3.4 Valuation of cashflows

So far we considered the valuation of a lump-sum random payment at time T. Now we generalize the theory to the pricing of payoff streams.

Specifically, we consider an adapted cashflow $h : [0,T] \to L^1(\mathcal{F}_T)$. We assume that h is Bochner integrable with respect to a finite measure μ on $[0,T]$ that weighs cashflows over time.

Given an equivalent martingale measure Q, the no arbitrage price process π of h is the expected discounted value of future cashflows under Q, i.e.

$$\pi(t) = \mathbb{E}_t^Q \left[\int_t^T e^{-r(\tau-t)} h_\tau \mu(d\tau) \right]$$

for all $t \in [0,T]$. For example, if μ is a counting measure, the previous formula evaluates a finite number or a sequence of future payments. In case μ is absolutely continuous, we are pricing instead a continuous stream of payoffs. In the next result we assume that μ is absolutely continuous with respect to the Lebesgue measure on $[0,T]$ and we write $\mu(dt) = p_t dt$, denoting the Radon-Nikodym derivative by p_t. We provide the risk-neutral pricing formula for cashflows in terms of weak time-derivatives.

Proposition 3.3 *Let $h \cdot p \in \mathcal{U}$ and $\pi_t = \mathbb{E}_t^Q[\int_t^T e^{-r(\tau-t)} h_\tau p_\tau d_\tau]$. Then, π belongs to $\mathcal{U}^1$ and solves the equation*

$$(\mathcal{D}\pi)_t = r\pi_t - h_t p_t \qquad \forall t \in [0,T].$$

Intuitively, if h_t is null except for the time T and μ has mass concentrated at T, we retrieve as special case the differential equation of Problem (3.1) about individual payoffs T. However, a formal claim about this case requires the theory of distributions and is beyond the scope of Proposition 3.3.

Moreover, observe that a term analogous to $-h_t p_t$ is added in the Feynman-Kac equation when a stream of dividends is present (see, e.g., Duffie [15], Appendix E).

Proof We denote $\mu(d\tau) = p_\tau d\tau$. We show that the process π of the statement is in $\mathcal{U}$ and it is weakly time-differentiable.

First, for all $\tau \in [0,T]$, $\pi_\tau \in L^1(\mathcal{F}_\tau)$ because h is Bochner integrable with respect to μ. Similarly, also the integral of $\mathbb{E}^Q[|\pi_\tau|]$ on $[0,T]$ is finite.

To establish L^1-continuity, fix any $t \in [0,T)$ and consider $\tau \to t^+$. By triangular inequality, $\mathbb{E}^Q[|\pi_\tau - \pi_t|]$ is smaller than

$$\mathbb{E}^Q\left[\left|e^{-r(t-\tau)}\mathbb{E}_\tau^Q\left[\int_\tau^T e^{-r(s-t)} h_s \mu(ds)\right] - \mathbb{E}_\tau^Q\left[\int_t^T e^{-r(s-t)} h_s \mu(ds)\right]\right|\right]$$
$$+ \mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[\int_t^T e^{-r(s-t)} h_s \mu(ds)\right] - \mathbb{E}_t^Q\left[\int_t^T e^{-r(s-t)} h_s \mu(ds)\right]\right|\right].$$

In the last sum, the second term converges to zero when τ approaches t by Lévy's Downward Theorem. The first term vanishes, too. Indeed, it is smaller than

$$\left|e^{-r(t-\tau)} - 1\right| \mathbb{E}^Q\left[\int_t^T e^{-r(s-t)} |h_s| \mu(ds)\right] + \mathbb{E}^Q\left[\int_t^\tau e^{-r(s-t)} |h_s| \mu(ds)\right].$$

The Bochner integrability of h ensures that these quantities are well-defined and convergent to zero as $\tau \to t^+$. A similar reasoning ensures the L^1-convergence from the left at T. So, π belongs to $\mathcal{U}$.

Now we compute the weak time-derivative of π. We consider any $A_t \in \mathcal{F}_t$ and any $\varphi \in C_c^1([t,T])$. The indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$ and so

$$-\int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\right]\varphi'(\tau)d\tau = -\int_t^T \mathbb{E}^Q\left[\int_\tau^T e^{-r(s-\tau)} h_s \mathbf{1}_{A_t} p_s ds\right]\varphi'(\tau)d\tau$$

because $\mu(ds) = p_s ds$. Since the expectation is a bounded operator, by Lemma 11.45 in Aliprantis and Border [2] we can exchange it with the integral. Along with integration by parts, we get:

$$
\begin{aligned}
&-\int_t^T \mathbb{E}^Q[\pi_\tau \mathbf{1}_{A_t}]\varphi'(\tau)d\tau \\
&= -\int_t^T \left(\int_\tau^T e^{-r(s-\tau)}\mathbb{E}^Q\left[h_s p_s \mathbf{1}_{A_t}\right] ds\right)\varphi'(\tau)d\tau \\
&= 0 + \int_t^T \frac{d}{d\tau}\left(\int_\tau^T e^{-r(s-\tau)}\mathbb{E}^Q\left[h_s p_s \mathbf{1}_{A_t}\right] ds\right)\varphi(\tau)d\tau \\
&= \int_t^T \left(-e^{-r(\tau-\tau)}\mathbb{E}^Q\left[h_\tau p_\tau \mathbf{1}_{A_t}\right] + r\int_\tau^T e^{-r(s-\tau)}\mathbb{E}^Q\left[h_s p_s \mathbf{1}_{A_t}\right] ds\right)\varphi(\tau)d\tau \\
&= \int_t^T \left(\mathbb{E}^Q\left[\left(-h_\tau p_\tau\right)\mathbf{1}_{A_t}\right] + \mathbb{E}^Q\left[r\int_\tau^T e^{-r(s-\tau)} h_s p_s \mathbf{1}_{A_t} ds\right]\right)\varphi(\tau)d\tau \\
&= \int_t^T \mathbb{E}^Q\left[\left(r\pi_\tau - h_\tau p_\tau\right)\mathbf{1}_{A_t}\right]\varphi(\tau)d\tau.
\end{aligned}
$$

Since both π and hp belong to $\mathcal{U}$, it follows that $r\pi - hp$ is included in $\mathcal{U}$. Therefore, the latter is the weak time-derivative of π. □

4 An operator approach

In this section we define the spaces and operators that allow us to formalize Problem (3.1) as an eigenvalue-eigenvector problem.

We first introduce some notation, still in the framework of Section 3. We denote by L_T the Radon-Nikodym derivative of the risk-neutral measure Q with respect to the physical measure P. Setting $L_t = \mathbb{E}_t[L_T]$ and $L_{t,T} = L_T/L_t$ for all $t \in [0,T]$, we are able to rewrite the no arbitrage price at time t as $\pi_t = e^{-r(T-t)}\mathbb{E}_t[L_{t,T}h_T]$. We can also restate the martingale property of discounted prices under Q by saying that the process $\{e^{-rt}L_t\pi(t)\}_{t\in[0,T]}$ is a martingale under the physical measure. In addition, the measure Q induces a stochastic discount factor process S that, at any $t \in [0,T]$, takes the form $S_t = e^{-rt}L_t$. See, for instance, Chapter 10 of Björk [6].

The starting point of our derivation is the observation that the no arbitrage pricing function π is weakly time-differentiable infinitely many times. Indeed, $\mathcal{D}\pi$ belongs to $\mathcal{U}$ and equals the original π except for the multiplicative constant r. Hence, $\mathcal{D}\pi$ is weakly time-differentiable, too. By defining the subspace

$$\mathcal{U}^\infty = \{u \in \mathcal{U} :\ u \text{ is infinitely weakly time-differentiable}\}$$

of $\mathcal{U}^1$, we can write $\pi \in \mathcal{U}^\infty$. The weak time-derivative defines a linear operator $\mathcal{D} : \mathcal{U}^\infty \to \mathcal{U}^\infty$ that maps any $u \in \mathcal{U}^\infty$ to $\mathcal{D}u$. Therefore, the differential equation of Problem (3.1) delivers the eigenvalue-eigenvector problem

$$\mathcal{D}\pi = r\pi, \qquad \pi \in \mathcal{U}^\infty. \tag{4.1}$$

This problem reformulates the one studied by Hansen and Scheinkman [21] who consider, instead of $\mathcal{D}$, the *extended generator* of the underlying Markov process. In our setting the Markov property is not required and the no arbitrage pricing function π is an eigenfunction of the operator $\mathcal{D}$ defined through weak time-derivatives. Moreover, $\{e^{-rt}L_t\pi(t)\}_t$ is a martingale process under P.

Following Hansen and Scheinkman [21], we choose a positive payoff h_T. The positivity of h_T is associated with the requirement of π to be an eigenfunction related to the *principal eigenvalue.* Indeed, Hansen and Scheinkman generalize the Perron-Frobenius theory (see Meyer [28], Chapter 8) from the finite-state Markov chain setting to more abstract frameworks.

Then, we define

$$\hat{L}_t = e^{-rt}L_t\frac{\pi_t}{\pi_0},$$

which still satisfies the martingale property. The stochastic discount factor S_t decomposes as

$$S_t = \hat{L}_t\frac{\pi_0}{\pi_t} = e^{-rt}\hat{L}_t\frac{\tilde{\pi}_0}{\tilde{\pi}_t},$$

where we define $\tilde{\pi}_t = \mathbb{E}_t[L_{t,T}h_T]$. In this decomposition, $-r$ is referred to as the *growth rate* of S_t, $\hat{L}_t$ is the *martingale component* and $\tilde{\pi}_0/\tilde{\pi}_t$ is the *transient component.* However, the decomposition is not unique.

This kind of results has proved to be fruitful in the macro-financial literature. For instance, Alvarez and Jermann [3] employ the last decomposition to quantify the dynamics of stochastic discount factors. Moreover, an application to the study of long-term risk-return trade-off for the valuation of cash flows is described in Hansen, Heaton, and Li [22].

The use of weak time-derivatives allows us to extend the applicability of Hansen-Scheinkman decomposition to a wide class of special semimartingales. The crucial point of the construction is the characterization of martingales through null weak time-derivatives.

Comparison with the infinitesimal generator

As we saw in Proposition 2.13, the weak time-derivative provides a way to differentiate random processes that generalizes the infinitesimal generator for Feller processes X and the extended infinitesimal generator for Markov processes. By focusing on the first one, if the infinitesimal generator of f is null, then the process $\{f(X_t)\}_{t\in[0,T]}$ is a martingale, a fact that parallels Proposition 2.4. Simple computations show that the no arbitrage pricing function of eq. (3.2) satisfies the eigenvalue-eigenvector problem $\mathcal{A}\pi = r\pi$. Hence, we can refer to $\mathcal{A}\pi = r\pi$ as a *strong form* eigenvalue-eigenvector problem, while Problem (3.1), rewritten as (4.1), defines a *generalized form.*

In addition, it holds that $\mathcal{A}(e^{-rt}\pi_t) = 0$, so the discounted price process $\{e^{-rt}\pi_t\}_{t\in[0,T]}$ is a martingale under Q. By exploiting the terminal condition $\pi_T = h_T$, this fact ensures that π is the unique solution of the problem in strong form.

We followed a parallel path of reasoning with weak time-derivatives, but with relevant differences: the class of processes involved and the continuity required (L^1 instead of uniform topology). A similar remark is valid for the extended infinitesimal generator.

5 No arbitrage pricing with stochastic interest rates

We provide a refinement of our theory to solve the no arbitrage pricing differential equation when interest rates are stochastic.

In this case we have two sources of randomness described by processes X and r defined on the probability space $(\Omega, \mathcal{F}, P)$ over the time interval $[0, T]$. As in Section 3, X is associated with the N-dimensional process of underlying stock prices. In addition, r is one-dimensional and represents stochastic instantaneous rates. Here, $B_t = e^{\int_0^t r_s ds}$ for all $t \in [0, T]$ and normalized prices are defined by $Z_t = e^{-\int_0^t r_s ds} X_t$. Hence, we consider the filtration generated by the pair (Z, r), which we denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$, assuming that the usual conditions hold.

In this section we assign a stronger meaning to weak time-differentiability. To distinguish the new definition from that of Section 2, we add L^∞ in front of weak. Indeed, L^∞*-weak time-differentiability* involves a larger set of test functions than weak time-differentiability of Definition 2.1. Specifically, we use as test functions all adapted processes in $C_c^1([t, T], L^\infty(\mathcal{F}_T))$. This space consists of the functions $\varphi : [0, T] \to L^\infty(\mathcal{F}_T)$ with compact support that are continuously differentiable in this sense: for any $t \in [0, T]$, $\varphi_t = \int_0^t \psi_\tau d\tau$ with $\psi : [0, T] \to L^\infty(\mathcal{F}_T)$ adapted, continuous and with compact support. In what follows we denote ψ by φ'.

Definition 5.1 We say that a process $u \in \mathcal{U}$ is L^∞*-weakly time-differentiable* when there exists a process $v \in \mathcal{U}$ such that, for every $t \in [0, T]$,

$$\int_t^T \mathbb{E}\left[v_\tau \mathbf{1}_{A_t} \varphi_\tau\right] d\tau = -\int_t^T \mathbb{E}\left[u_\tau \mathbf{1}_{A_t} \varphi'_\tau\right] d\tau$$

for all $A_t \in \mathcal{F}_t$ and all adapted $\varphi \in C_c^1([t, T], L^\infty(\mathcal{F}_T))$. In this case, we call v a L^∞*-weak time-derivative* of u.

We denote by $\mathcal{U}_\infty^1$ the space of L^∞-weakly time-differentiable $u \in \mathcal{U}$. Definition 5.1 is well-posed because the integrals therein are finite for any choice of A_t and φ. Indeed, φ and φ' are continuous functions that take values in $L^\infty(\mathcal{F}_T)$, so their image is bounded.

If u is L^∞-weakly time-differentiable, it is also weakly time-differentiable because the test functions φ, which are random processes, may specialize to deterministic functions. Hence, we inherit some of the results of Section 2. For instance, the L^∞-weak time-derivative is still unique. Moreover, if a process $u \in \mathcal{U}$ is L^∞-weakly time-differentiable with $\mathcal{D}u = 0$, then it is a martingale.

In our financial application, we assume that instantaneous rates define an adapted process $r : [0,T] \to L^\infty(\mathcal{F}_T)$. In addition, we impose that interest rates are uniformly bounded over time - i.e. there is a positive $\tilde{R}$ such that $|r_t| \leqslant \tilde{R}$ for all $t \in [0,T]$ - and that they are L^2-right-continuous in any $t \in [0,T)$ and L^2-left-continuous at T. Progressive measurability (which holds up to modifications) and boundedness ensure the Bochner integrability of r. As a result, the Bochner integral $\int_0^T r_\tau d\tau$ is a well-defined object in $L^\infty(\mathcal{F}_T)$.

By following the same line of reasoning of Section 3, from NFLVR we infer the existence of a sigma-martingale measure Q, which we assume to be an equivalent martingale measure. Thus, we move to the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. Since now interest rates are stochastic, under the measure Q the no arbitrage pricing differential equation is

$$\begin{cases} (\mathcal{D}\pi)_t = r_t \pi_t & t \in [0,T) \\ \pi_T = h_T \end{cases} \tag{5.1}$$

where we assume $h_T \in L^2(\mathcal{F}_T)$. Differently from Problem (3.1), now each r_t belongs to $L^\infty(\mathcal{F}_t)$ and $\mathcal{D}$ represents the L^∞-weak time-derivative. We now show the unique solution of this problem in $\mathcal{U}^1_\infty$.

Theorem 5.2 *Under the previous assumptions on r, there exists a unique solution π of Problem* (5.1) *in $\mathcal{U}^1_\infty$, given by*

$$\pi_t = \mathbb{E}^Q_t \left[e^{-\int_t^T r_s ds} h_T \right] \qquad \forall t \in [0,T].$$

Proof Existence.

We prove that π belongs to $\mathcal{U}$ and it is L^∞-weakly time-differentiable.

First, for all $\tau \in [0,T]$, $\pi_\tau \in L^1(\mathcal{F}_\tau)$ because r is uniformly bounded and $h_T \in L^1(\mathcal{F}_T)$. In addition, the integral of $\mathbb{E}^Q[|\pi_\tau|]$ over $[0,T]$ is finite.

As to L^1-continuity, for all $t \in [0,T)$, consider $\tau \to t^+$. Then, by triangular inequality,

$$\begin{aligned} \mathbb{E}^Q[|\pi_\tau - \pi_t|] &\leqslant \mathbb{E}^Q \left[e^{-\int_t^T r_s ds} |h_T| \left| e^{-\int_\tau^t r_s ds} - 1 \right| \right] \\ &\quad + \mathbb{E}^Q \left[\left| \mathbb{E}^Q_\tau \left[e^{-\int_t^T r_s ds} h_T \right] - \mathbb{E}^Q_t \left[e^{-\int_t^T r_s ds} h_T \right] \right| \right] \\ &\leqslant e^{(T-t)\tilde{R}} \mathbb{E}^Q \left[|h_T| \left| e^{-\int_\tau^t r_s ds} - 1 \right| \right] \\ &\quad + \mathbb{E}^Q \left[\left| \mathbb{E}^Q_\tau \left[e^{-\int_t^T r_s ds} h_T \right] - \mathbb{E}^Q_t \left[e^{-\int_t^T r_s ds} h_T \right] \right| \right] \end{aligned}$$

because r is uniformly bounded by $\tilde{R}$. Moreover, we can apply Lagrange's Theorem to the continuously differentiable function $\sigma \mapsto e^{-\int_\sigma^t r_s ds}$ for all σ in the interval $[t,\tau]$. Hence, we find $\hat{t} \in (t,\tau)$ such that

$$e^{-\int_\tau^t r_s ds} - 1 = r_{\hat{t}} e^{-\int_{\hat{t}}^t r_s ds} (\tau - t). \tag{5.2}$$

As a result,

$$e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|e^{-\int_\tau^t r_s ds}-1\right|\right] = e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|r\left(\hat{t}\right)\right|e^{-\int_{\hat{t}}^t r_s ds}\right](\tau-t)$$
$$\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\right]\tilde{R}e^{(t-\hat{t})\tilde{R}}(\tau-t).$$

Consequently,

$$\mathbb{E}^Q\left[|\pi_\tau-\pi_t|\right] \leqslant e^{(T-\hat{t})\tilde{R}}\mathbb{E}^Q\left[|h_T|\right](\tau-t)\tilde{R}$$
$$+\,\mathbb{E}^Q\left[\left|\mathbb{E}_\tau^Q\left[e^{-\int_t^T r_s ds}h_T\right]-\mathbb{E}_t^Q\left[e^{-\int_t^T r_s ds}h_T\right]\right|\right].$$

Both terms on the right tend to zero as τ reaches t^+ (the convergence of the second one is guaranteed by Lévy's Downward Theorem). A parallel argument ensures the L^1-convergence at T from the left. Therefore, π belongs to $\mathcal{U}$.

Now we look for the L^∞-weak time-derivative of π. We take any $A_t \in \mathcal{F}_t$ and any adapted $\varphi \in C_c^1([t,T], L^\infty(\mathcal{F}_T))$. Recall that indicator functions $\mathbf{1}_{A_t}$ are $\mathcal{F}_\tau$-measurable for all $\tau \in [t,T]$. Since φ' is adapted too, we deduce that

$$-\int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\varphi'_\tau\right]d\tau = -\int_t^T \mathbb{E}^Q\left[e^{-\int_\tau^T r_s ds}h_T\mathbf{1}_{A_t}\varphi'_\tau\right]d\tau.$$

$e^{-\int_\tau^T r_s ds}\varphi'_\tau$ is a continuous function of $\tau \in [t,T]$, hence it is Bochner integrable. The expectation is a bounded operator, so Lemma 11.45 in Aliprantis and Border [2] allows us to exchange expectation and integral. Therefore,

$$-\int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\varphi'_\tau\right]d\tau = -\mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T e^{-\int_\tau^T r_s ds}\varphi'_\tau d\tau\right]$$
$$= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(1-e^{-\int_\tau^T r_s ds}\right)\varphi'_\tau d\tau\right] - \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T \varphi'_\tau d\tau\right]$$
$$= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(1-e^{-\int_\tau^T r_s ds}\right)\varphi'_\tau d\tau\right]$$

because φ has compact support. Now consider the function $u \mapsto r_u e^{-\int_u^T r_s ds}$. This function is Bochner integrable (because r is uniformly bounded) and its Bochner integral coincides almost surely with the pathwise Lebesgue integral:

$$\int_\tau^T r_u e^{-\int_u^T r_s ds}du = 1-e^{-\int_\tau^T r_s ds}.$$

By exploiting integration by parts (see Craven [11]), we obtain

$$
\begin{aligned}
&-\int_t^T \mathbb{E}^Q\left[\pi_\tau \mathbf{1}_{A_t}\varphi'_\tau\right]d\tau = \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T\left(\int_\tau^T r_u e^{-\int_u^T r_s ds}du\right)\varphi'_\tau d\tau\right]\\
&= \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t}\int_t^T r_\tau e^{-\int_\tau^T r_s ds}\varphi_\tau d\tau\right] = \int_t^T \mathbb{E}^Q\left[h_T\mathbf{1}_{A_t} r_\tau e^{-\int_\tau^T r_s ds}\varphi_\tau\right]d\tau\\
&= \int_t^T \mathbb{E}^Q\left[r_\tau\pi_\tau\mathbf{1}_{A_t}\varphi_\tau\right]d\tau.
\end{aligned}
$$

Therefore, the candidate L^∞-weak time-derivative of π is $r\pi$ and $r_t\pi_t$ belongs to $L^1(\mathcal{F}_t)$ for all t because r is bounded. As for L^1-continuity, let τ go to t^+ for any $t\in[0,T)$. Then, by triangular inequality,

$$
\begin{aligned}
\mathbb{E}^Q\left[\left|r_\tau\pi_\tau - r_t\pi_t\right|\right] \leqslant{}& \mathbb{E}^Q\left[\left|r_\tau e^{-\int_\tau^T r_s ds}h_T - r_t e^{-\int_t^T r_s ds}h_T\right|\right]\\
&+ \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r_t e^{-\int_t^T r_s ds}h_T\right] - \mathbb{E}^Q_t\left[r_t e^{-\int_t^T r_s ds}h_T\right]\right|\right].
\end{aligned}
$$

By exploiting the uniform boundedness of r and eq. (5.2), we find that the first addend in the last expression satisfies

$$
\begin{aligned}
&\mathbb{E}^Q\left[\left|r_\tau e^{-\int_\tau^T r_s ds}h_T - r_t e^{-\int_t^T r_s ds}h_T\right|\right]\\
&\quad\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|r_\tau e^{-\int_\tau^t r_s ds} - r_t\right|\right]\\
&\quad= e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|r_\tau + r_\tau r_{\hat{t}} e^{-\int_{\hat{t}}^t r_s ds}(\tau - t) - r_t\right|\right]\\
&\quad\leqslant e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|r_\tau - r_t\right|\right] + e^{(T-t)\tilde{R}}\mathbb{E}^Q\left[|h_T|\left|r_\tau r_{\hat{t}}\right| e^{-\int_{\hat{t}}^t r_s ds}\right]|\tau - t|.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathbb{E}^Q\left[\left|r_\tau\pi_\tau - r_t\pi_t\right|\right] \leqslant{}& e^{(T-t)\tilde{R}}\left(\mathbb{E}^Q\left[h_T^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}^Q\left[|r_\tau - r_t|^2\right]\right)^{\frac{1}{2}}\\
&+ e^{(T-\hat{t})\tilde{R}}\tilde{R}^2\mathbb{E}^Q\left[|h_T|\right]|\tau - t|\\
&+ \mathbb{E}^Q\left[\left|\mathbb{E}^Q_\tau\left[r_t e^{-\int_t^T r_s ds}h_T\right] - \mathbb{E}^Q_t\left[r_t e^{-\int_t^T r_s ds}h_T\right]\right|\right].
\end{aligned}
$$

As τ approaches t^+, the first term goes to zero because $h_T\in L^2(\mathcal{F}_T)$ and r is L^2-right-continuous; the second term tends to zero because r is uniformly bounded; the last term is convergent to zero by Lévy's Downward Theorem. Therefore, the L^1-right-continuity is proved. Analogous steps guarantee the L^1-left-continuity at T. Hence, $r\pi$ belongs to $\mathcal{U}$ and it is the L^∞-weak time-derivative of π. Summing up, $\pi\in\mathcal{U}^1_\infty$ and solves Problem (5.1).

Uniqueness.

Let $\pi,\hat{\pi}\in\mathcal{U}^1_{L^\infty}$ be two solutions of Problem (5.1). By defining $z=\pi-\hat{\pi}$ in $\mathcal{U}^1_\infty$, we have that, for every $t\in[0,T)$, $(\mathcal{D}z)_t = r_t z_t$ and $z_T = 0$.

The process r is Bochner integrable over time. Reasoning state by state, we have $\int_t^T r_s ds = R_T - R_t$, where R is the random process that collects state-by-state primitives of r.

Now we determine the L^∞-weak time-derivative of the process $e^{-R_t} z_t$. For any adapted $\varphi \in C_c^1([t,T], L^\infty(\mathcal{F}_T))$, consider the function

$$u \mapsto e^{-R_u} r_u \varphi_u - e^{-R_u} \varphi'_u.$$

Since r is bounded, this function is Bochner integrable. By reasoning pathwise,

$$\int_\tau^T \left(e^{-R_u} r_u \varphi_u - e^{-R_u} \varphi'_u\right) du = e^{-R_\tau} \varphi_\tau.$$

Hence, $e^{-R}\varphi$ is adapted, it belongs to $C_c^1([t,T], L^\infty(\mathcal{F}_T))$ and so we can use it as test function in the definition of L^∞-weak time-derivative of z:

$$\int_t^T \mathbb{E}^Q \left[(\mathcal{D}z)_\tau \, \mathbf{1}_{A_t} e^{-R_\tau} \varphi_\tau \right] d\tau = - \int_t^T \mathbb{E}^Q \left[z_\tau \mathbf{1}_{A_t} \left(e^{-R_\tau} \varphi'_\tau - e^{-R_\tau} r_\tau \varphi_\tau \right) \right] d\tau.$$

Rearranging terms, we obtain that the L^∞-weak time-derivative of $e^{-R_t} z_t$ is $e^{-R_t}((\mathcal{D}z)_t - r_t z_t)$. Since this process is null, $e^{-R_t} z_t$ has null L^∞-weak time-derivative. Therefore, by following the proof of Proposition 2.4 for test functions in $C_c^1([t,T], L^\infty(\mathcal{F}_T))$, $e^{-R_t} z_t$ constitutes a martingale. Then, for every $t \in [0,T]$ and $\tau \in [t,T]$

$$\mathbb{E}_t^Q \left[e^{-R_\tau} z_\tau \right] = e^{-R_t} z_t.$$

As τ approaches T from the left, $\mathbb{E}_t^Q[e^{-R_\tau} z_\tau]$ goes to zero in L^1 because e^{-R_τ} is bounded and z_τ tends to $z_T = 0$ in L^1. So, by uniqueness of the L^1-limit, $e^{-R_t} z_t = 0$. As a result, $z_t = 0$ for all $t \in [0,T]$ and uniqueness of the solution of Problem (5.1) is established. □

6 Conclusions

We introduced the weak time-derivative, a novel mathematical tool that allows us to differentiate stochastic processes in a more general way than infinitesimal generators and permits to best profit from standard calculus insights. It provides easy characterizations of martingales and permits to formulate differential equations for random processes in weak form. We expect that weak time-derivatives will be useful for different kinds of problems, beyond the ones discussed in this work.

A fruitful application of the weak time-derivative involves the solution of the no arbitrage pricing equation for random payoffs. In particular, the generalized form that we solve clarifies the central role of interest rates in driving the asset prices. In addition, constant interest rates deliver an eigenvalue-eigenvector formulation of the risk-neutral pricing equation in full agreement with the long-term risk literature. Nevertheless, how to set up the analogous eigenvalue-eigenvector problem when interest rates are stochastic still remains an open problem. Indeed, the candidate eigenvalue would be a random process itself. Moreover, such a formulation should be able to generate a term structure of interest rates.

Another promising direction of research comes from the analysis of price dynamics through the lenses of different risk-neutral measures, associated with specific numéraire changes (see Geman, El Karoui, and Rochet [19]). Indeed, the risk-free rate r is an eigenvalue when the measure Q is employed. Under a different equivalent martingale measure (as, for instance, the forward measure in the context of stochastic rates), the first question to answer is whether a differential equation with the structure of eq. (3.1) is still valid. A plausible possibility is that the same dynamics are present, but the driving parameter is not r. Hence, the second question is the identification of the proper eigenvalue according to the employed measure. This approach, made possible by the flexibility of weak time-derivatives, opens a promising perspective on payoff valuation in various contexts, while remaining within the boundaries of no arbitrage theory.

The research questions presented above require several notions from fixed-income markets. Severino [34] attempts to address them by using weak time-derivatives to analyse the impact of the aggregation of short-term rate risks on cashflows valuation, over longer and longer maturities.

A Appendix: a routine lemma

We state and prove a version of a standard result that is best suited for our purposes.

Lemma A.1 *Let $f : [t, T] \to \mathbb{R}$ be a measurable function.*

(i) If f is bounded, nonnegative, with compact support and $\int_t^T f(\tau)g(\tau)d\tau = 0$ for any $g \in C_c([t, T])$, then $f = 0$ a.e.
(ii) If $\int_t^T f(\tau)g(\tau)d\tau = 0$ for any $g \in C_c([t, T])$, then $f = 0$ a.e.

Proof (*i*) If f is strictly positive on a set A with positive measure, consider the indicator function $\mathbf{1}_A$ and a sequence $\{U_n\}_n$ of continuous positive approximations of $\mathbf{1}_A$, obtained by convolution with a smooth positive kernel. As U_n converges to $\mathbf{1}_A$ in L^2, $0 \leqslant \int_t^T f(\tau)\mathbf{1}_A(\tau)d\tau = \lim_n \int_t^T f(\tau)U_n(\tau)d\tau = 0$. In consequence, f is null a.e.

(*ii*) Suppose that f is positive with compact support. For any $N > 0$ consider $f_N(s) = \min\{f(\tau), N\}$. Then,

$$0 \leqslant \int_t^T f_N(\tau)g(\tau)d\tau \leqslant \int_t^T f(\tau)g(\tau)d\tau = 0.$$

Therefore, each f_N is null a.e. by (i) and so f is. □

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